Improving algebraic tools to study bifurcation sequences of population models

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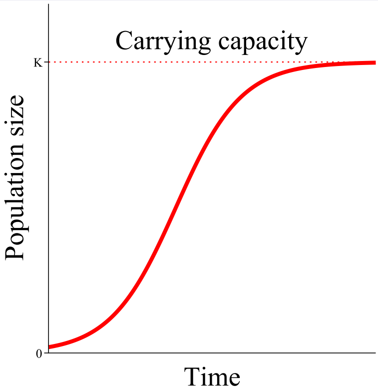
Contents

- The question of interest in population dynamics.
- Conversion into language of Algebraic Geometry.
- Some former existing algorithms and their limits.
- A new algorithm.
- Conclusion.

Population growth

Logistic growth



$$\dot{N} = rN(1 - \frac{N}{K})$$

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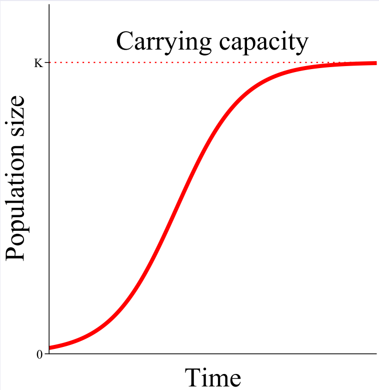

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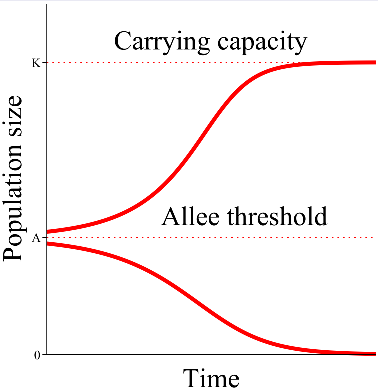
(Strong) Allee effect



$$\dot{N} = rN(1 - \frac{N}{K})(\frac{N}{A} - 1)$$

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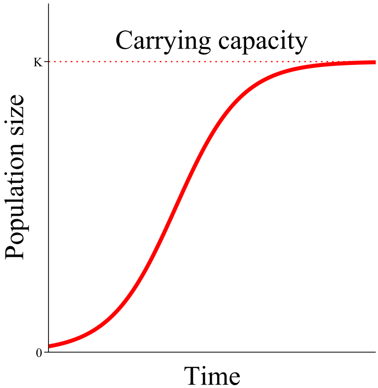

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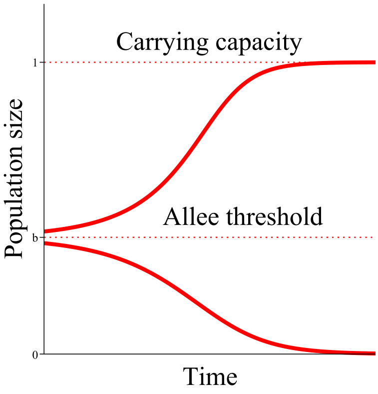
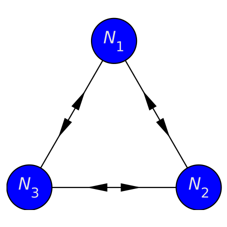


$$\dot{N} = N(1 - N)(N - b)$$

Main model

n-connected populations with Allee effect.

$$\dot{N}_i = N_i(1 - N_i)(N_i - b) - (n-1)aN_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} aN_j, \quad i = 1, \cdots, n.$$



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$$N_i(1 - N_i)(N_i - b) - (n-1)aN_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} aN_j = 0, \quad i = 1, \cdots, n.$$

This parametric system has n variables, N_is, and 2 parameters, the Allee effect parameter b and the strength of connectivity a.

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Questions

- What happens for not very small and not very large values of a?
- Is it true that the number of non-negative solutions is non-increasing with respect to a?

Approach no. 1

Original system
$\{x^2 + bx + c = 0\}$

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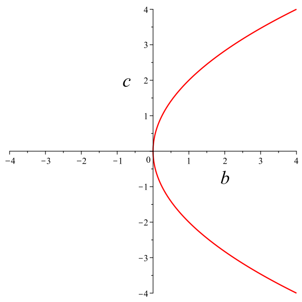

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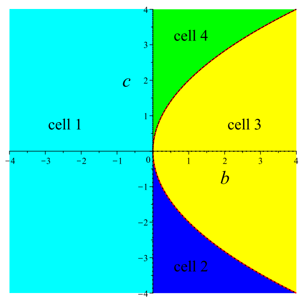

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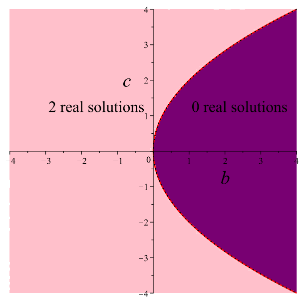
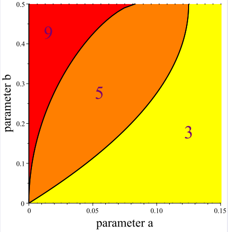
$\#(f^{-1}(0) \cap \mathbb{R}) =$
$\begin{cases} 2 & ;\ (c, b) \in \text{cells 1,2,4} \\ 0 & ;\ (c, b) \in \text{cell 3} \end{cases}$



Using approach 1 (used in ref. 3)

Remember the *n*-patches model.

Maple* can compute the 2-dimensional CAD* of the model for $n = 2$.



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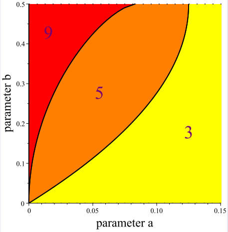

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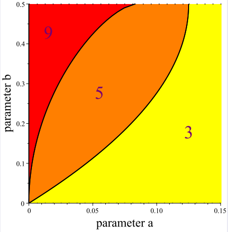
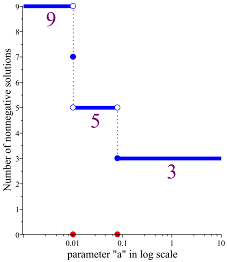


By fixing the value of b, Maple* can compute the 1-dimensional CAD* of the model for $n = 2, 3, 4$, but not $n = 5$.

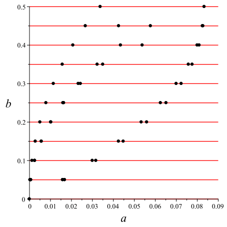
As an example the case $n = 2$ for $b = 0.2$ is shown here.



Approach no. 2 (introduced and used in ref. 2)

Using a numeric search.

Finding sections of the 2-dimensional CAD using 1-dimensional CADs for some finite samples of a parameter.



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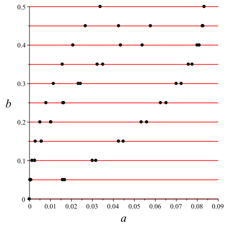

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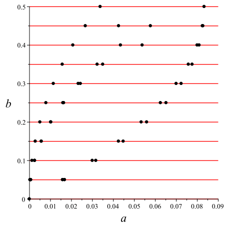

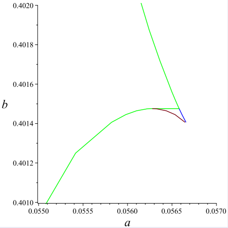

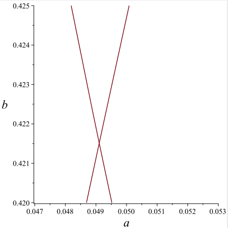

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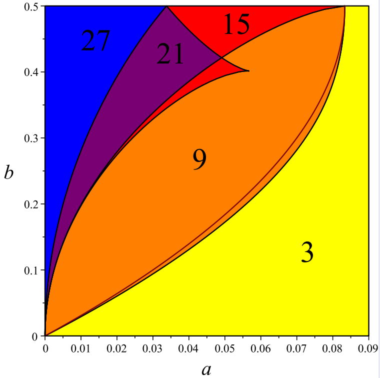
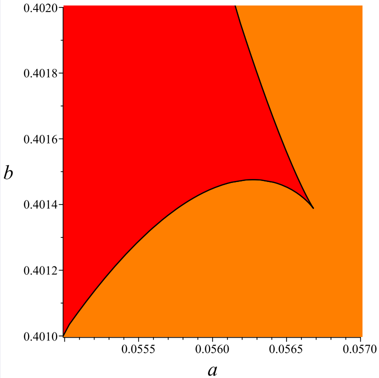
Up to 7 digits accuracy after the decimal point





A discovery

The number of steady states is not always decreasing monotonically by increasing *a*.

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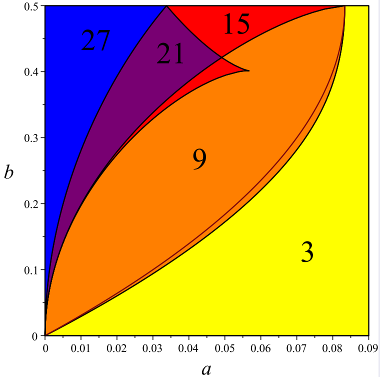

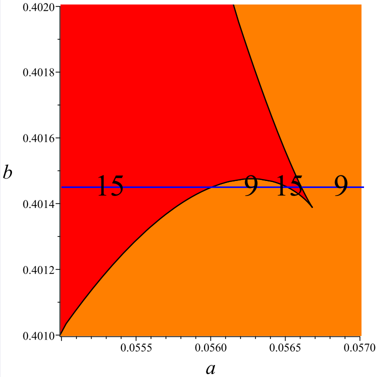

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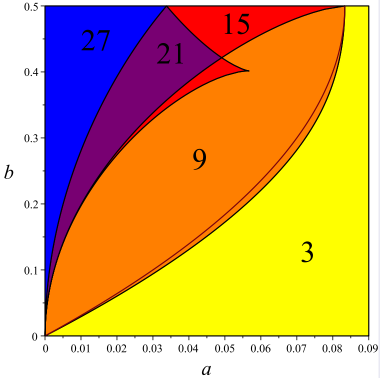
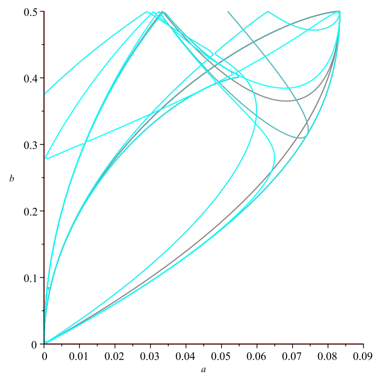
3. Using Equational Constraints (ECs) the computations in the previous step becomes singly exponential.
4. An open CAD with respect to the new variety also answers the questions of our interest.

Implemented by Maple

Comparision





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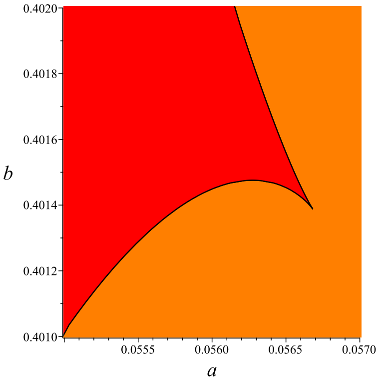

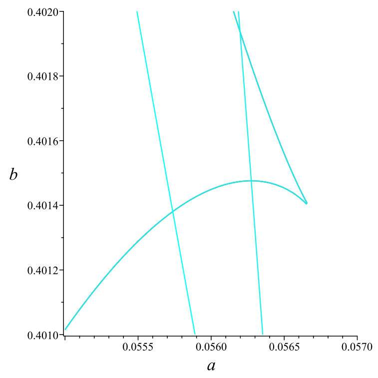

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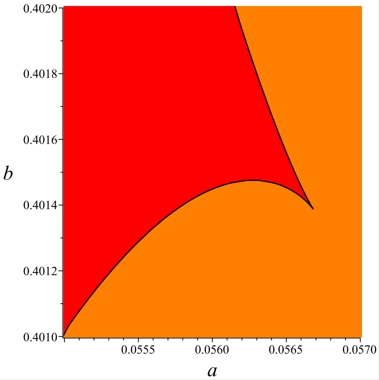

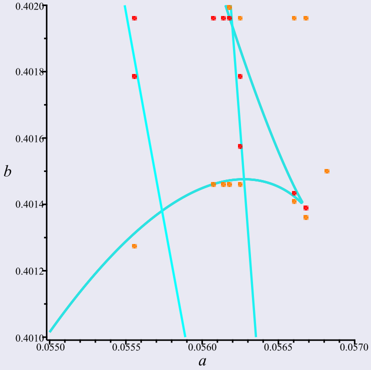

What is new?

We implemented a new algorithm in Maple that can handle larger size examples of parametric system of equations (and inequalities) with the following properties;

1. It is free of Gröbner bases computation.
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Does it mean that the former numerical approach is not interesting anymore?
No, one still can equip approach 2 on top of approach 3 and go even further in the size of examples that can be handled by a normal computer.

References

1. AmirHosein Sadeghimanesh, Matthew England, *Improving algebraic tools to study bifurcation sequences of population models*, in progress, 2021.
2. Gergely Röst, AmirHosein Sadeghimanesh, *Exotic bifurcations in three connected populations with Allee effects*, bioArxiv, 2021, DOI: 10.1101/2021.02.03.429609.
3. Gergely Röst, AmirHosein Sadeghimanesh, *Unidirectional migration of populations with Allee effect*, bioArxiv, 2021, DOI: 10.1101/2021.06.24.449708.

Thank you for listening.