Building Abelian Functions with Generalised Hirota Operators

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(based on a joint paper with Chris Athorne)

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Outline

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Abelian functions associated to algebraic curves

We consider functions periodic w.r.t. the period lattice Λ of an algebraic curve of genus g. If ω_1, ω_2 are the period matrices then

$$\Lambda = \{\omega_1 \boldsymbol{m} + \omega_2 \boldsymbol{n} \,|\, m, n \in \mathbb{Z}^g\}.$$

Definition

Let $\mathfrak{M}(\boldsymbol{u})$ be a meromorphic function of $(u_1, u_2, \ldots, u_g) = \boldsymbol{u} \in \mathbb{C}^g$. Then $\mathfrak{M}(\boldsymbol{u})$ is an Abelian function associated with the curve if

$$\mathfrak{M}(\boldsymbol{u} + \omega_1 \boldsymbol{n} + \omega_2 \boldsymbol{m}) = \mathfrak{M}(\boldsymbol{u}),$$

for all integer vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$ where $\mathfrak{M}(\boldsymbol{u})$ is defined.

The simplest case are the elliptic functions, when $g = 1$.

Kleinian $\wp$-functions

Kleinian $\wp$-functions

Define the Kleinian $\wp$-functions as the second log derivatives of the curve's multivariate σ-function, $\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, ..., u_g)$:

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\mathbf{u}), \quad i \leq j \in \{1, 2, ..., g\}.$$

We can extend this notation to higher order derivatives:

$$\wp_{i_1,\dots,i_m} = -\frac{\partial^m}{\partial u_{i_1} \dots \partial u_{i_m}} \ln \sigma(\mathbf{u}), \qquad i_1 \leq \cdots \leq i_m \in \{1, 2, ..., g\}.$$

They are all Abelian functions and those with m indices have poles of order m on the Θ-divisor.

Bases of Abelian functions

Let $\Gamma(m)$ be the vector space of Abelian functions with poles of order at most m occurring only on the Θ-divisor.
We seek bases for such spaces.

- The Riemann-Roch theorem for Abelian varieties gives $\dim(\Gamma(m)) = g^m$.
- We start by including the basis for $\Gamma(m-1)$. We then need to find $g^m - g^{m-1}$ linearly independent functions with poles of order exactly m.
- The m-index Kleinian $\wp$-functions may be included.

Example: Genus one case

Space	Dim	Basis
$\Gamma(1)$	1	1
$\Gamma(2)$	2	$1, \wp$
$\Gamma(3)$	3	$1, \wp, \wp'$
$\Gamma(4)$	4	$1, \wp, \wp', \wp''$
$\vdots$	$\vdots$	$\vdots$
$\Gamma(m)$	m	$1, \wp, \wp', \wp'', \dots, \wp^{(m-2)}$

A table of bases for the elliptic case: Here $\wp$ is the Weierstrass elliptic $\wp$-function.

Example: Genus two case

Space	Dim	Basis for $\Gamma(m)\backslash\Gamma(m-1)$
$\Gamma(1)$	1	1
$\Gamma(2)$	4	$\wp_{11}, \wp_{12}, \wp_{22}$
$\Gamma(3)$	9	$\wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}, \Delta$
$\Gamma(4)$	16	$\wp_{1111}, \dots, \wp_{2222}, \partial_1\Delta, \partial_2\Delta$
$\Gamma(5)$	25	$\wp_{11111}, \dots, \wp_{22222}, \partial_{11}\Delta, \partial_{12}\Delta, \partial_{22}\Delta$
$\vdots$	$\vdots$	$\vdots$
$\Gamma(m)$	m^2	$\{\wp_{i_1\dots i_m}\}, \{\partial_{i_1}\dots\partial_{i_{m-2}}\Delta\}$

A table of bases for the genus two case: Here we use genus two Kleinian $\wp$-functions and the function $\Delta = \wp_{11}\wp_{22} - \wp_{12}^2$ along with its derivatives. We use ∂_i for differentiation with respect to u_i and $\{\cdot\}$ to denote all functions of that form.

The basis problem

- We know the dimension of $\Gamma(m)$. For (n, s)-curves at least, there are well developed techniques to test the linear independence of elements using the σ-expansion and weight arguments. The basis problem is hence the identification of enough suitable functions of a given pole order.
- For $g > 2$ we need more than just $\wp$'s, Δ and their derivatives. New functions have been defined to build bases for many specific cases. These are usually polynomials in $\wp$-functions defined through a pole matching procedure, analogous to Δ.

Examples of basis problem solutions

- In EEMOP07 basis for $\Gamma(3)$ in (3,4)-case found using

$$\wp^{[ij]} := \text{determinant of the } (i,j) - \text{minor of } [\wp_{ij}]_{3\times 3}.$$

- In EEO11 basis for $\Gamma(3)$ in (2,7)-case found using $\wp^{[ij]}$ and

$$T = 2\wp_{22}^3 + \wp_{222}^2 - \wp_{22}\wp_{2222}.$$

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- However, for $g > 2$ hyperelliptic and $g > 3$ non-hyperelliptic, the bases are not finitely generated by differentiation, [Nakayashiki,Cho]. So infinitely many new functions are required.

The Hirota operator and $\wp$-functions

We can define Hirota's bilinear operator as

$$\mathcal{D}_i = \frac{\partial}{\partial u_i} - \frac{\partial}{\partial v_i}.$$

We may then check that

$$\wp_{ij}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2} \mathcal{D}_i \mathcal{D}_j \sigma(\boldsymbol{u}) \sigma(\boldsymbol{v}) \Big|_{\boldsymbol{v}=\boldsymbol{u}}.$$

The Q-functions

The Q-functions

We define the *n-index Q-functions* as

$$Q_{i_1,i_2,\ldots,i_n}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2}\mathcal{D}_{i_1}\mathcal{D}_{i_2}\ldots\mathcal{D}_{i_n}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\;\Big|_{\boldsymbol{v}=\boldsymbol{u}}$$

- So $Q_{ij} = \wp_{ij}$. But further indices denote application of operators, not differentiation.
- Note that is applied with n odd then they are identically zero.
- First Q-function used by Baker. The 4-index functions defined in EEMOP07. Six index functions required for theory of non-hyperelliptic genus six curves.

Bases of $\Gamma(2)$

All Q-functions are Abelian functions with poles of order two and so can be used to construct bases of $\Gamma(2)$.

- Genus 3 hyperelliptic: $\{1, \wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}, Q_{2222}\}$.
- Genus 3 trigonal: $\{1, \wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}, Q_{1333}\}$.
- Genus 6 tetragonal: $\{\wp_{11}, \dots, \wp_{66}, Q_{5566}, \dots, Q_{1144}, Q_{114466}\}$.

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- Genus 6 tetragonal: $\{\wp_{11}, \dots, \wp_{66}, Q_{5566}, \dots, Q_{1144}, Q_{114466}\}$.

The choice of functions is not unique. For instance, in the trigonal example we may use Q_{2222} instead of Q_{1333}. Also, in the hyperelliptic example we could instead use

$$\Delta_B = \wp_{11}\wp_{33} - \wp_{12}\wp_{23} - \wp_{13}^2 + \wp_{13}\wp_{22}.$$

Aim: To define something similar to Q-functions, which can be used to complete bases for $\Gamma(m)$ where $m > 2$.

Outline

1. Motivation
   - Abelian functions and the basis problem
   - The Hirota operator
2. Generalised Hirota operators and $\mathcal{R}$-functions
   - Definitions
   - Proofs and examples

Changing notation to tensor products

The original Hirota derivatives act on pairs, not products.

The Hirota operator is bilinear over addition and scalar multiplication, so we can treat the pairs as tensor products. The original Hirota derivative is then a composition of the Hirota operators with a total symmetrization operator, S, mapping from the tensor to the differential polynomial algebra:

$$\begin{array}{ccccc} F \otimes F & \xrightarrow{\mathcal{D}_i} & F \otimes F & \xrightarrow{S} & F, \\ f \otimes g & \xmapsto{\mathcal{D}_i} & f_i \otimes g - f \otimes g_i & \xmapsto{S} & f_i g - f g_i. \end{array}$$

where F denotes a k-algebra of appropriate k-valued functions, for some field k. In this talk we let $k = \mathbb{C}$.

Generalised Hirota operators

We now work with the tensor product of m multivariate functions.

$$\bigotimes_{k=1}^{m} f^{[k]} = f^{[1]} \otimes \cdots \otimes f^{[m]}.$$

The variables are $\boldsymbol{u} = (u_1, \ldots, u_g)$. Define the operator $\partial_i^{[j]}$ by

$$\partial_i^{[j]} (f^{[1]} \otimes \cdots \otimes f^{[j]} \otimes \cdots \otimes f^{[m]}) = f^{[1]} \otimes \cdots \otimes f_i^{[j]} \otimes \cdots \otimes f^{[m]}.$$

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Generalised Hirota operators

Define $\mathcal{H}_i^{[m]}$ to be the mth order generalised Hirota operator which acts on the tensor product of m functions as

$$\mathcal{H}_i^{[m]} \left(\bigotimes_{k=1}^{m} f^{[k]} \right) = \sum_{j=1}^{m} \zeta^{j-1} \, \partial_i^{[j]} \bigotimes_{k=1}^{m} f^{[k]},$$

where ζ is a primitive mth root of unity.

Connection with previous results

When $m = 2$ we have

$$\mathcal{H}_i^{[2]}(f \otimes g) = f_i \otimes g - f \otimes g_i$$

The total symmetrization operator acts as

$$S\left(\bigotimes_{k=1}^{m} f^{[k]}\right) = \prod_{k=1}^{m} f^{[k]}.$$

So we see,

$$Q_{i_1,\dots,i_n} = -\frac{1}{2\sigma^2} S \circ \mathcal{H}_{i_1}^{[2]} \circ \cdots \circ \mathcal{H}_{i_n}^{[2]}(\sigma \otimes \sigma).$$

$\mathcal{R}$-functions

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Define the *n-index mth order $\mathcal{R}$-functions* as

$$\mathcal{R}^{[m]}_{i_1,\dots,i_n} = \left(-\frac{1}{m\sigma^m}\right)\left(S \circ \mathcal{H}^{[m]}_{i_1} \circ \cdots \circ \mathcal{H}^{[m]}_{i_n} \bigotimes_{k=1}^{m} \sigma\right).$$

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- The functions clearly have poles of order m. We find they are also Abelian and so can be used to construct bases of $\Gamma(m)$.
- We find that they are identically zero unless $m|n$.
- We have,

$$Q_{i_1,\ldots,i_n} = \mathcal{R}^{[2]}_{i_1,\ldots,i_n}$$
$$\wp_{i_1,\ldots,i_n} = \mathcal{R}^{[n]}_{i_1,\ldots,i_n}$$

Sketch proof of periodicity I

1. The σ-function is quasi-periodic. For $\boldsymbol{\ell} \in \Lambda$ we have $\sigma(\boldsymbol{u} + \boldsymbol{\ell}) = h\sigma(\boldsymbol{u})$ where $h = \chi e^{L(\boldsymbol{u})}$ and L is s.t. all $L_{ij} = 0$.

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2. Define a multiplication operation between tensor products as

$$\bigotimes_{k=1}^{m} f^{[k]} \cdot \bigotimes_{k=1}^{m} g^{[k]} = \bigotimes_{k=1}^{m} f^{[k]} g^{[k]}.$$

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3. The Hirota operators satisfy a corresponding product rule,

$$\mathcal{H}_i^{[m]}\left(\bigotimes_{k=1}^{m} f^{[k]} \cdot \bigotimes_{k=1}^{m} g^{[k]}\right) = \bigotimes_{k=1}^{m} f^{[k]} \cdot \mathcal{H}_i^{[m]}\left(\bigotimes_{k=1}^{m} g^{[k]}\right) + \bigotimes_{k=1}^{m} g^{[k]} \cdot \mathcal{H}_i^{[m]}\left(\bigotimes_{k=1}^{m} f^{[k]}\right).$$

Sketch proof of periodicity II

4. There is a corresponding Leibniz Rule,

$$\mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \left(\bigotimes_{k=1}^{m} f^{[k]} \right) \cdot \left(\bigotimes_{k=1}^{m} g^{[k]} \right)$$
$$= \sum_{\ell=0}^{n} \sum_{\pi\in\Pi} \mathcal{H}_{\pi_1}^{[m]} \left(\bigotimes_{k=1}^{m} f^{[k]} \right) \cdot \mathcal{H}_{\pi_2}^{[m]} \left(\bigotimes_{k=1}^{m} g^{[k]} \right).$$

Π the set of disjoint partitions, π, of the indices $\{i_1, \dots, i_n\}$ into two subsets π_1 and π_2 of lengths $n-\ell$ and ℓ. $\mathcal{H}_{\pi_i}^{[m]}$ denotes concatenation of generalised Hirota operators.

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5. We may show inductively that for all n

$$S \circ \mathcal{H}^{[m]}_{i_1} \circ \cdots \circ \mathcal{H}^{[m]}_{i_n} \bigotimes_{k=1}^{m} h = 0.$$

Sketch proof of periodicity III

So putting it all together,

$$\mathcal{R}^{[m]}_{i_1,\dots,i_n}(\boldsymbol{u}+\boldsymbol{\ell}) = \left(-\frac{1}{m(h\sigma)^m}\right)\left(S\circ\mathcal{H}^{[m]}_{i_1}\circ\cdots\circ\mathcal{H}^{[m]}_{i_n}\bigotimes_{k=1}^{m} h\sigma\right) \quad \text{by 1}$$

$$= \left(-\frac{1}{mh^m\sigma^m}\right)\left(S\circ\mathcal{H}^{[m]}_{i_1}\circ\cdots\circ\mathcal{H}^{[m]}_{i_n}\bigotimes_{k=1}^{m} h\cdot\bigotimes_{k=1}^{m}\sigma\right) \quad \text{by 2}$$

$$= \left(-\frac{1}{mh^m\sigma^m}\right)\left(S\circ\sum_{\ell=0}^{n}\sum_{\pi\in\Pi}\mathcal{H}^{[m]}_{\pi_1}\left(\bigotimes_{k=1}^{m} h\right)\cdot\mathcal{H}^{[m]}_{\pi_2}\left(\bigotimes_{k=1}^{m}\sigma\right)\right) \quad \text{by 4}$$

$$= \left(-\frac{1}{mh^m\sigma^m}\right)\left(h^m\cdot S\circ\mathcal{H}^{[m]}_{i_1}\circ\cdots\circ\mathcal{H}^{[m]}_{i_n}\bigotimes_{k=1}^{m}\sigma\right) \quad \text{by 5}$$

$$= \mathcal{R}^{[m]}_{i_1,\dots,i_n}(\boldsymbol{u}).$$

Sketch proof of identically zero properties I

1. We observe that

$$\mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} f^{[k]} = \sum_{\rho \in P_n^m} \sum_{\psi \in \Psi(\rho)} \zeta^{z(\psi)} \sum_{\pi \in \Pi(\psi)} \bigotimes_{k=1}^{m} f_{\pi_k}^{[k]}$$

where ρ are partitions of n, ψ permutations of ρ and π set partitions of $\{i_1, \ldots, i_n\}$ into subsets of lengths given by ψ.

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   where ρ are partitions of n, ψ permutations of ρ and π set partitions of $\{i_1, \ldots, i_n\}$ into subsets of lengths given by ψ.

2. Then, with $X_k = \zeta^{k-1}$ we can show

$$S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} f = \sum_{\rho \in P_n^m} M_\rho(X) \left(\sum_{\pi \in \Pi(\rho)} \left(\prod_{k=1}^{m} f_{\pi_k} \right) \right).$$

   Here $M_\rho(x)$ are monomial symmetric functions. E.g.

$$M_{[2,1,0]}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

Sketch proof of identically zero properties II

3. For $X_k = \zeta^{k-1}$ the power sum symmetric functions are

$$p_k = \begin{cases} m & \text{if } m|k \\ 0 & \text{otherwise} \end{cases}$$

Sketch proof of identically zero properties II

3. For $X_k = \zeta^{k-1}$ the power sum symmetric functions are

$$p_k = \begin{cases} m & \text{if } m|k \\ 0 & \text{otherwise} \end{cases}$$

4. All symmetric functions may be expressed as a polynomial in $p_1, \ldots, p_m$. Using the degree of polynomials as a grading, we see that unless $m|n$, then such expressions for $M_\rho(x)$ will be zero, and hence $\mathcal{R}^{[m]}_{i_1,\ldots,i_n}$ also.

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5. If $n|m$ then $\hat{\rho} = [n, 0, \ldots, 0] \in P^m_n$ and $M_{\hat{\rho}}(X) = p_n(X) \neq 0$, so $\mathcal{R}^{[m]}_{i_1,\ldots,i_n}$ is not identically zero in these cases.

Sketch proof that all Kleinian $\wp$-functions are $\mathcal{R}$-functions

1. We introduce $\hat{\wp}$ such that $\sigma = e^{\hat{\wp}}$. Then $\wp_{i_1,\dots,i_n} = -\hat{\wp}_{i_1,\dots,i_n}$.

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2. So applying a Hirota operator;

$$\mathcal{H}_{i_1}^{[m]} \bigotimes_{k=1}^{m} \sigma = \sum_{j=1}^{m} \zeta^{j-1} \partial_i^{[j]} \bigotimes_{k=1}^{m} e^{\hat{\wp}}$$

$$= \sum_{j=1}^{m} \zeta^{j-1} (1 \otimes \cdots \otimes \hat{\wp}_{i_1} \otimes \cdots \otimes 1) \left(\bigotimes_{k=1}^{m} e^{\hat{\wp}} \right) =: \Sigma_1 \left(\bigotimes_{k=1}^{m} \sigma \right)$$

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3. Applying another Hirota operator to Σ_1 will give many terms, but all except one will evaluate to zero upon application of S.

$$\mathcal{H}_{i_2}^{[m]} (\Sigma_1) \doteq \sum_{j=1}^{m} \zeta^{2(j-1)} (1 \otimes \cdots \otimes \hat{\wp}_{i_1 i_2} \otimes \cdots \otimes 1) =: \Sigma_2$$

Sketch proof that all Kleinian $\wp$-functions are $\mathcal{R}$-functions

4. So we have

$$\mathcal{H}_{i_1}^{[m]} \circ \mathcal{H}_{i_2}^{[m]} \bigotimes_{k=1}^{m} \sigma \doteq \Sigma_2 \left(\bigotimes_{k=1}^{m} \sigma \right) + \Sigma_1 \mathcal{H}_{i_2}^{[m]} \left(\bigotimes_{k=1}^{m} \sigma \right).$$

Sketch proof that all Kleinian $\wp$-functions are $\mathcal{R}$-functions

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5. In general,

$$\mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_m}^{[m]} \bigotimes_{k=1}^{m} \sigma \doteq \Sigma_m \left(\bigotimes_{k=1}^{m} \sigma \right) + \ldots$$

where $\Sigma_m = \sum_{j=1}^{m} \zeta^{m(j-1)} \left(1 \otimes \cdots \otimes \hat{\wp}_{i_1,\ldots,i_m} \otimes 1 \otimes 1\right)$

and all other terms, have as a factor, a sequence of $\mathcal{H}$ of size less than m, applied to a tensor product of σ. Hence under S,

$$S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_m}^{[m]} \bigotimes_{k=1}^{m} \sigma = m \hat{\wp}_{i_1,\ldots,i_m} \sigma^m.$$

So $\mathcal{R}_{i_1,i_2,\ldots,i_m}^{[m]}(\boldsymbol{u}) = \wp_{i_1,i_2,\ldots,i_m}(\boldsymbol{u})$.

Example: Bases for hyperelliptic genus 3 curve: (2,7)-curve

$\Gamma(2)$:

$$\{1, \mathcal{R}^{[2]}_{11}, \mathcal{R}^{[2]}_{12}, \mathcal{R}^{[2]}_{13}, \mathcal{R}^{[2]}_{22}, \mathcal{R}^{[2]}_{23}, \mathcal{R}^{[2]}_{33}, \mathcal{R}^{[2]}_{2222}\}.$$

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$\Gamma(3)\backslash\Gamma(2)$:

$$\left\{\begin{array}{ccccccc} \mathcal{R}^{[3]}_{111}, & \mathcal{R}^{[3]}_{122}, & \mathcal{R}^{[3]}_{222}, & \mathcal{R}^{[3]}_{333}, & \mathcal{R}^{[3]}_{233333}, & \mathcal{R}^{[3]}_{222222}, & \partial_1\mathcal{R}^{[2]}_{2222}, \\ \mathcal{R}^{[3]}_{112}, & \mathcal{R}^{[3]}_{123}, & \mathcal{R}^{[3]}_{223}, & & \mathcal{R}^{[3]}_{133333}, & \mathcal{R}^{[3]}_{113333}, & \partial_2\mathcal{R}^{[2]}_{2222}, \\ \mathcal{R}^{[3]}_{113}, & \mathcal{R}^{[3]}_{133}, & \mathcal{R}^{[3]}_{233}, & & \mathcal{R}^{[3]}_{222333}, & \mathcal{R}^{[3]}_{112333}, & \partial_3\mathcal{R}^{[2]}_{2222} \end{array}\right\}.$$

$\Gamma(4)\backslash\Gamma(3)$:

$$\left\{\begin{array}{ccccccc} \mathcal{R}^{[4]}_{1111}, & \mathcal{R}^{[4]}_{1133}, & \mathcal{R}^{[4]}_{2222}, & \partial_3\mathcal{R}^{[3]}_{233333}, & \partial_1\mathcal{R}^{[3]}_{133333}, & \partial_2\mathcal{R}^{[3]}_{112333}, & \mathcal{R}^{[4]}_{12333333}, \\ \mathcal{R}^{[4]}_{1112}, & \mathcal{R}^{[4]}_{1222}, & \mathcal{R}^{[4]}_{2223}, & \partial_2\mathcal{R}^{[3]}_{233333}, & \partial_2\mathcal{R}^{[3]}_{222333}, & \partial_3\mathcal{R}^{[3]}_{222222}, & \mathcal{R}^{[4]}_{11333333}, \\ \mathcal{R}^{[4]}_{1113}, & \mathcal{R}^{[4]}_{1223}, & \mathcal{R}^{[4]}_{2233}, & \partial_3\mathcal{R}^{[3]}_{133333}, & \partial_1\mathcal{R}^{[3]}_{222333}, & \partial_1\mathcal{R}^{[3]}_{112333}, & \mathcal{R}^{[4]}_{11233333}, \\ \mathcal{R}^{[4]}_{1122}, & \mathcal{R}^{[4]}_{1233}, & \mathcal{R}^{[4]}_{2333}, & \partial_1\mathcal{R}^{[3]}_{233333}, & \partial_2\mathcal{R}^{[3]}_{113333}, & \partial_2\mathcal{R}^{[3]}_{222222}, & \mathcal{R}^{[4]}_{12223333}, \\ \mathcal{R}^{[4]}_{1123}, & \mathcal{R}^{[4]}_{1333}, & \mathcal{R}^{[4]}_{3333}, & \partial_2\mathcal{R}^{[3]}_{133333}, & \partial_1\mathcal{R}^{[3]}_{113333}, & \partial_1\mathcal{R}^{[3]}_{222222}, & \mathcal{R}^{[4]}_{11133333}, \\ & & & & & & \mathcal{R}^{[4]}_{22222222}, \\ & & & & & & \mathcal{R}^{[4]}_{11123333}. \end{array}\right\}.$$

Example: Bases for trigonal genus 3 curve: (3,4)-curve

$\Gamma(2)$:

$$\{1, \mathcal{R}^{[2]}_{11}, \mathcal{R}^{[2]}_{12}, \mathcal{R}^{[2]}_{13}, \mathcal{R}^{[2]}_{22}, \mathcal{R}^{[2]}_{23}, \mathcal{R}^{[2]}_{33}, \mathcal{R}^{[2]}_{2222}\}.$$

$\Gamma(3)\backslash\Gamma(2)$:

$$\left\{ \begin{array}{lllllll} \mathcal{R}^{[3]}_{111}, & \mathcal{R}^{[3]}_{122}, & \mathcal{R}^{[3]}_{222}, & \mathcal{R}^{[3]}_{333}, & \mathcal{R}^{[3]}_{333333}, & \mathcal{R}^{[3]}_{113333}, & \partial_1\mathcal{R}^{[2]}_{2222}, \\ \mathcal{R}^{[3]}_{112}, & \mathcal{R}^{[3]}_{123}, & \mathcal{R}^{[3]}_{223}, & & \mathcal{R}^{[3]}_{222333}, & \mathcal{R}^{[3]}_{122333}, & \partial_2\mathcal{R}^{[2]}_{2222}, \\ \mathcal{R}^{[3]}_{113}, & \mathcal{R}^{[3]}_{133}, & \mathcal{R}^{[3]}_{233}, & & \mathcal{R}^{[3]}_{133333}, & \mathcal{R}^{[3]}_{122233}, & \partial_3\mathcal{R}^{[2]}_{2222} \end{array} \right\}.$$

$\Gamma(4)\backslash\Gamma(3)$:

$$\left\{ \begin{array}{lllllll} \mathcal{R}^{[4]}_{1111}, & \mathcal{R}^{[4]}_{1133}, & \mathcal{R}^{[4]}_{2222}, & \partial_3\mathcal{R}^{[3]}_{333333}, & \partial_2\mathcal{R}^{[3]}_{133333}, & \partial_2\mathcal{R}^{[3]}_{122233}, & \mathcal{R}^{[4]}_{22333333}, \\ \mathcal{R}^{[4]}_{1112}, & \mathcal{R}^{[4]}_{1222}, & \mathcal{R}^{[4]}_{2223}, & \partial_2\mathcal{R}^{[3]}_{333333}, & \partial_3\mathcal{R}^{[3]}_{122333}, & \partial_2\mathcal{R}^{[3]}_{113333}, & \mathcal{R}^{[4]}_{22233333}, \\ \mathcal{R}^{[4]}_{1113}, & \mathcal{R}^{[4]}_{1223}, & \mathcal{R}^{[4]}_{2233}, & \partial_3\mathcal{R}^{[3]}_{222333}, & \partial_1\mathcal{R}^{[3]}_{222333}, & \partial_1\mathcal{R}^{[3]}_{122333}, & \mathcal{R}^{[4]}_{13333333}, \\ \mathcal{R}^{[4]}_{1122}, & \mathcal{R}^{[4]}_{1233}, & \mathcal{R}^{[4]}_{2333}, & \partial_1\mathcal{R}^{[3]}_{333333}, & \partial_2\mathcal{R}^{[3]}_{122333}, & \partial_1\mathcal{R}^{[3]}_{122233}, & \mathcal{R}^{[4]}_{12233333}, \\ \mathcal{R}^{[4]}_{1123}, & \mathcal{R}^{[4]}_{1333}, & \mathcal{R}^{[4]}_{3333}, & \partial_2\mathcal{R}^{[3]}_{222333}, & \partial_1\mathcal{R}^{[3]}_{133333}, & \partial_1\mathcal{R}^{[3]}_{113333}, & \mathcal{R}^{[4]}_{12223333}, \\ & & & & & & \mathcal{R}^{[4]}_{11333333}, \\ & & & & & & \mathcal{R}^{[4]}_{11223333}. \end{array} \right\}.$$

Conjectures arising from examples

The two genus three cases have *structurally similar bases*. I.e. The same number of functions of each *type* used in corresponding bases. Not apparent when same bases constructed in EEO11.

Conjecture 1: Bases of Abelian functions associated with curves of the same genus share a structure.

Conjectures arising from examples

The two genus three cases have *structurally similar bases*. I.e. The same number of functions of each *type* used in corresponding bases. Not apparent when same bases constructed in EEO11.

Conjecture 1: Bases of Abelian functions associated with curves of the same genus share a structure.

It appears that $\Gamma(m)\backslash\Gamma(m-1)$ may always be filled using mth order $\mathcal{R}$-functions and first derivatives of $(m-1)$th order $\mathcal{R}$-functions.

Conjecture 2: Abelian functions are spanned by $\mathcal{R}$ and $\partial\mathcal{R}$.

Further Information

M. England, C. Athorne
Building Abelian functions with generalised Hirota operators.
Preprint: arXiv:1203.3409 (2012)

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