Elliptic Curve Cryptography

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Outline

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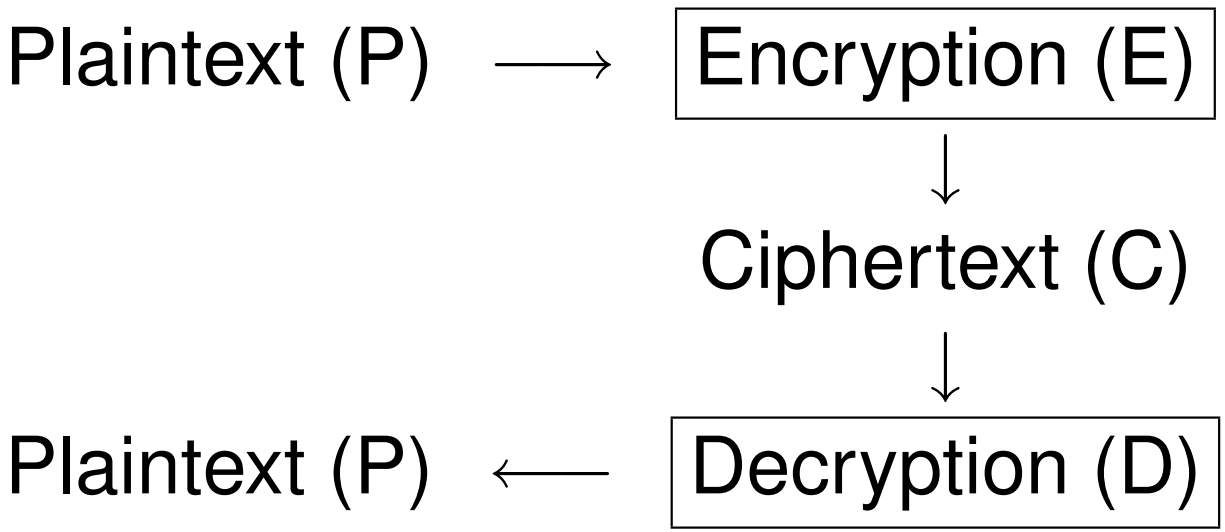
Cryptosystems

Cryptosystems are constructed to securely send messages.



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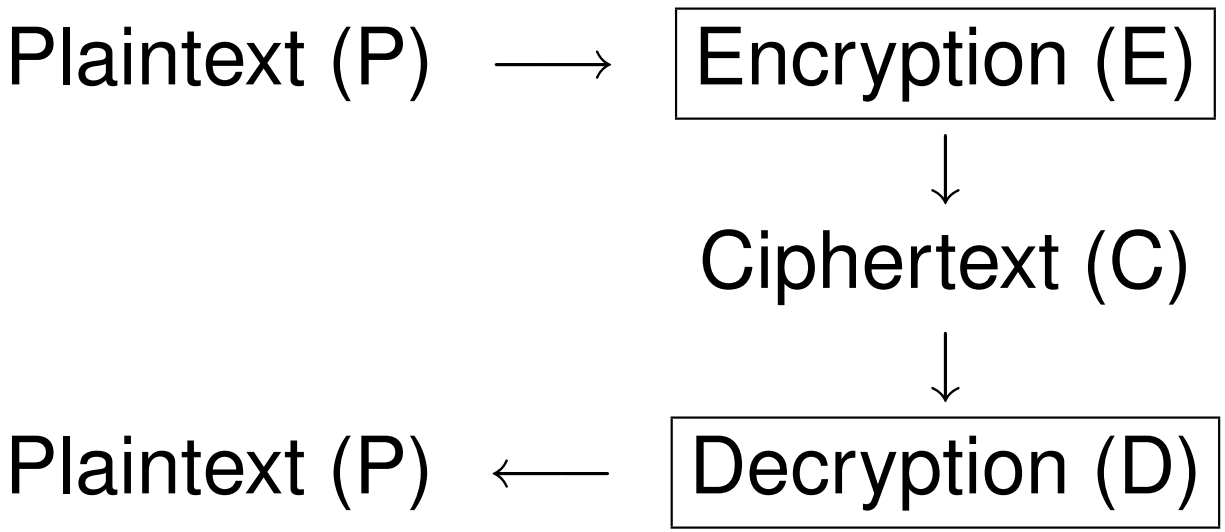

- P and C are often just bit streams (0s and 1s).

What is cryptography?

Definition

CRYPTOGRAPHY is the science of keeping messages secure.

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CRYPTANALYSIS are the techniques for breaking ciphertext.

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+

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CRYPTANALYSIS are the techniques for breaking ciphertext.

= CRYPTOLOGY

This is all more important now that ever!

Development of cryptography

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Two main types of cryptosystems:

Symmetric Key Systems

- Decryption key $=$ Encryption key
- Users need to agree on this key **before** communicating securely.

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Example: The Caesar Cipher

In this simple cipher each letter is moved 3 places to the right

$$\text{eg} \quad \text{CRYPTOGRAPHY} \longrightarrow \text{FUBSWRJUDSKB}$$

This cipher could be broken with a minimal amount of effort!

- However, there are much more sophisticated systems regularly used in the real world (eg DES, AES).

Two main types of cryptosystem

Symmetric Key Systems

- Decryption key $=$ Encryption key
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OR

Public Key Systems

- Decryption key $\neq$ Encryption key
- Encryption Key $=$ Public
  Decryption Key $=$ Private

A Public Key Encryption system

Private:  **Public**: 

A Public Key Encryption system

Private:  **Public**: 

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Alice ⟶ Plaintext

↓

Encryption  Bob's Public Key

↓

Ciphertext

↓

Decryption  Bob's Private Key

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Bob ⟵ Plaintext

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In the above diagram:

- No keys exchanged!
- Bob is the only person who could read this message.

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- First suggested by Diffie and Hellman (1976)
- First put into practise by RSA (1978)
- Other notable schemes include El Gamal (1984)

A Public Key System can be used for authentication as well as encryption...

A Public Key Authentication system

Private:  **Public**: 

A Public Key Authentication system

Private:  **Public:** 

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Alice ⟶ Plaintext

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Encryption ←  Alice's Private Key

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Decryption ←  Alice's Public Key

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A Public Key Authentication system

Private:  **Public**: 

Alice ⟶ Plaintext

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Encryption  Alice's Private Key

↓

Ciphertext

↓

Decryption  Alice's Public Key

↓

Bob ⟵ Plaintext

- Only Alice could have possibly sent the message!

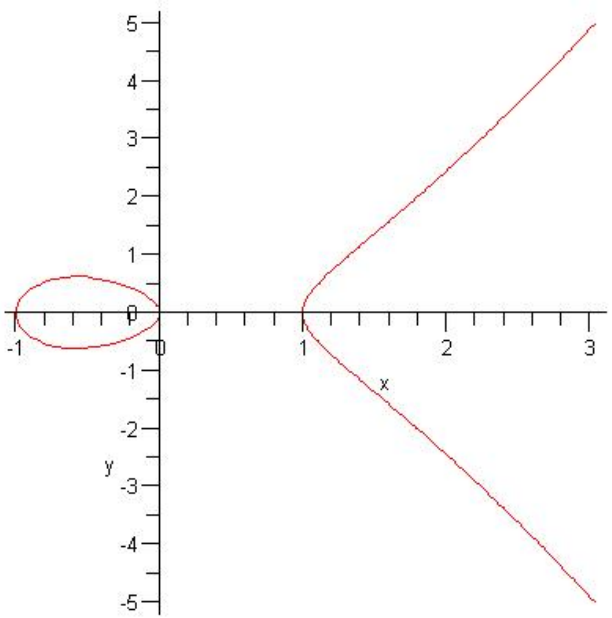
Outline

What is an elliptic curve?

Definition

An elliptic curve is a (non-singular) algebraic curve defined by an equation of the form

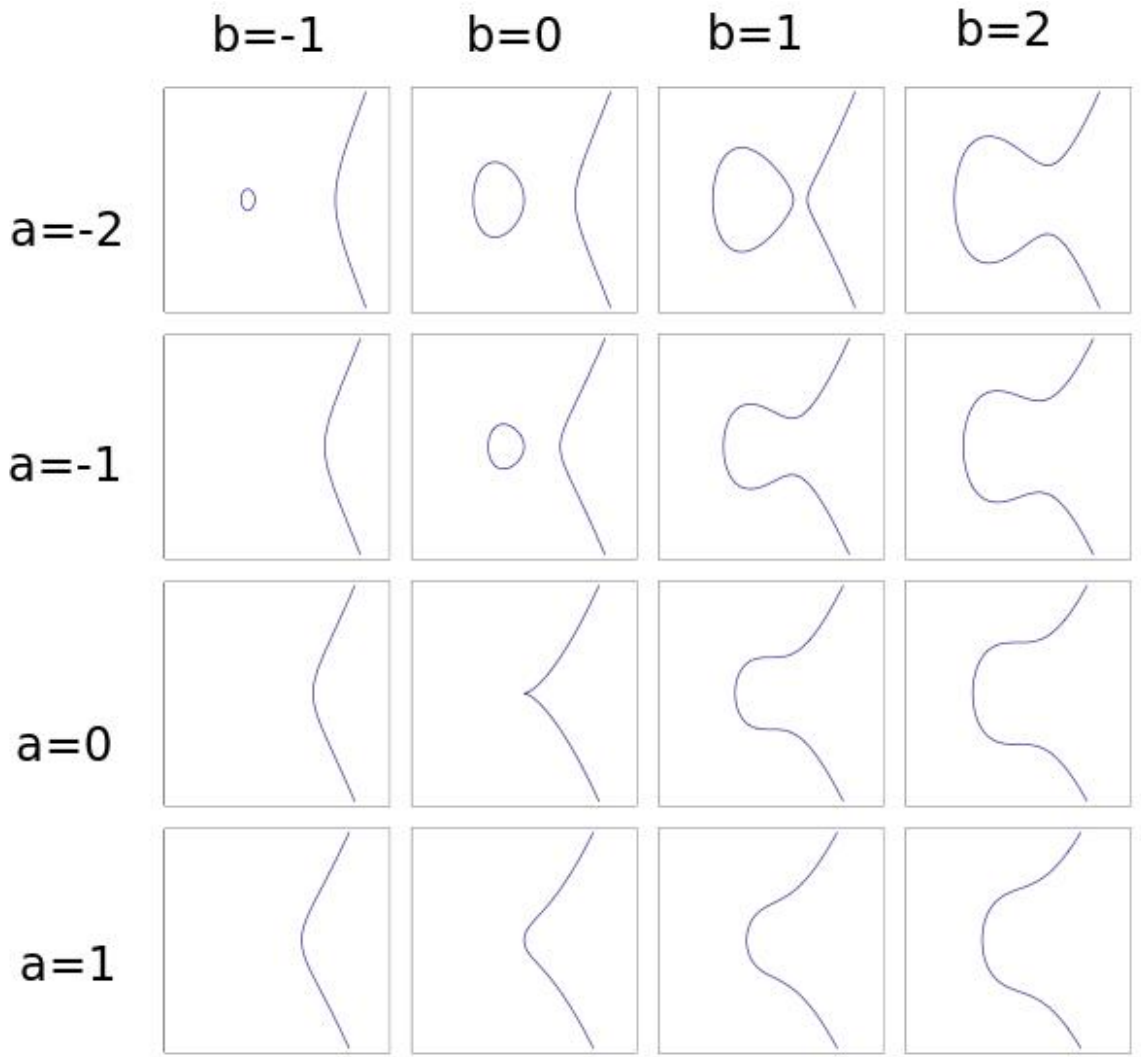
$$Y^2 = X^3 + aX + b$$



For example, when
$a = -1$ and $b = 0$ we have:

$$Y = X^3 - X$$

More examples of elliptic curves



Defining an addition operation

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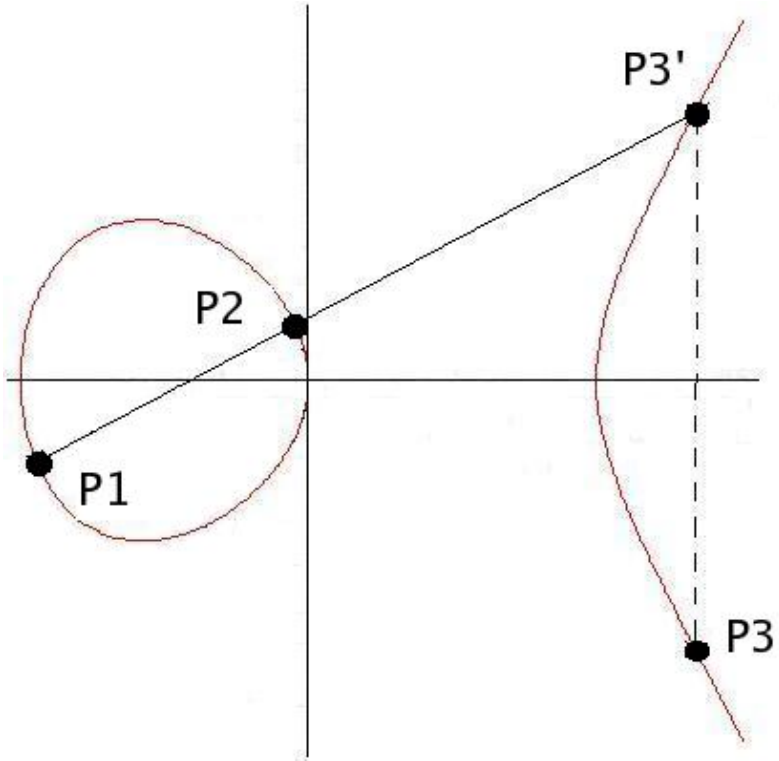
Given two points $P1$ and $P2$:

1. Find the straight line connecting them.
2. Calculate the third point of intersection $P3'$.
3. Reflect to find $P3$.

Define the addition law as

$$P1 + P2 = P3$$



▸ Infinity

Sorting out the exceptions

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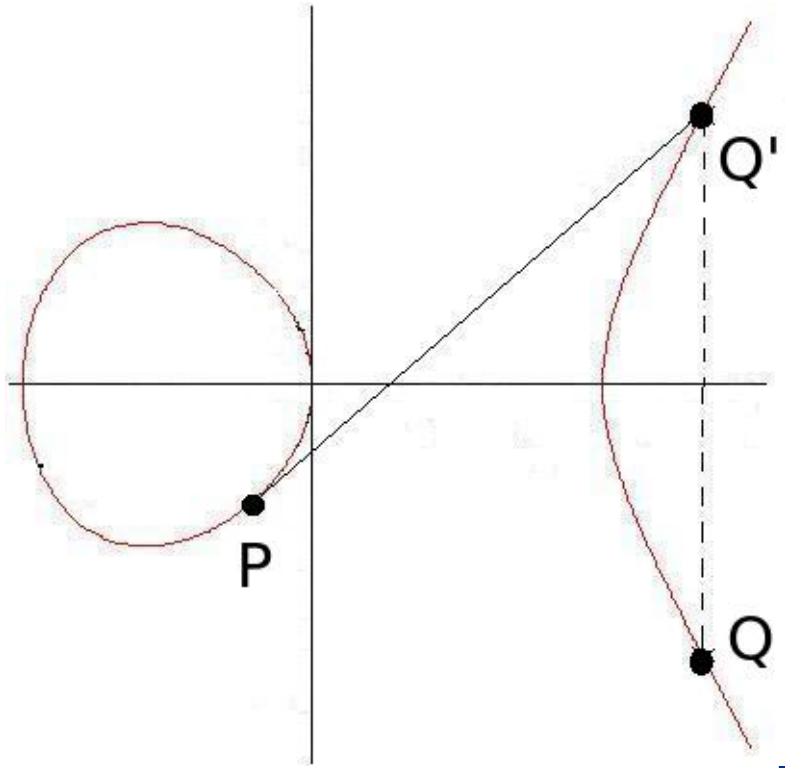
A) Instead of drawing the line connecting the two points, draw the tangent line to the curve at the point.
Then, as before, find the third point of intersection, Q' and reflect to find the point Q. Define $P + P = Q$.



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- Think of ∞ as sitting at both the top (and bottom) of the y-axis — then these rules coincide with the process above.
- The point ∞ lies upon the elliptic curve.

For more information study projective geometry!

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Points on an elliptic curve form a group!

Consider an operation $*$ and a set of objects, G. These form a group $(G, *)$ if the four axioms below are satisfied.

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The point ∞ satisfies the identity property. ▸ infinity

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  Clear from the definition of the addition operation.

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Why can elliptic curves be used in cryptography?

Work with an elliptic curve defined over a finite field.

Definition

Let k be a positive integer, and P a point on an elliptic curve. Then multiplying P by k is defined as

$$kP = P + P + P + ... + P \qquad (k \text{ times})$$

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- Finding Q given k, P — Easy! (Use successive doubling)
- Finding k given Q, P — Difficult , often infeasible!

$\Longrightarrow$ one-way function, ideal for Public Key Cryptography!

Real World

The Diffie-Hellman Key Exchange for elliptic curves

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A spy would know E, P, P_a and P_b. Finding abP from this information is infeasible!

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Bob sets up the cryptosystem by choosing an elliptic curve, E, a point, P and a secret integer s. He computes $B = sP$ and makes, E, P and B public, (keeping s private).

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2. Alice chooses a secret integer r, and computes

$$M_1 = rP, \qquad \text{and} \quad M_2 = M + rB$$

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- RSA has so far dominated the use of Public Key cryptography...
- Elliptic curve systems give a similar level of security to RSA while using a much smaller bit size!

Further Reading

Simon Singh.
The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography.
Fourth Estate, 1999.

Steven Levy.
Crypto: How the Code Rebels Beat the Government - Saving Privacy in the Digital Age.
Chapman and Hall/CRC, 2003.

Lawrence C. Washington.
Elliptic Curves: Number Theory and Cryptography.
Chapman and Hall/CRC, 2003.