

Improving algebraic tools to study bifurcation sequences of population models

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(Joint work with Matthew England)

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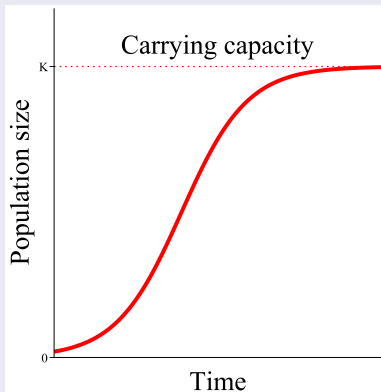
Maple Conference 2021

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- The question of interest in population dynamics.
- Conversion into language of Algebraic Geometry.
- Some former existing algorithms and their limits.
- A new algorithm.
- Conclusion.

Population growth

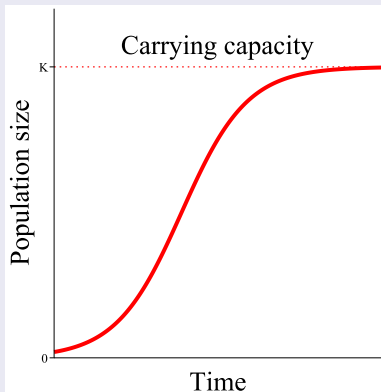
Logistic growth



$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

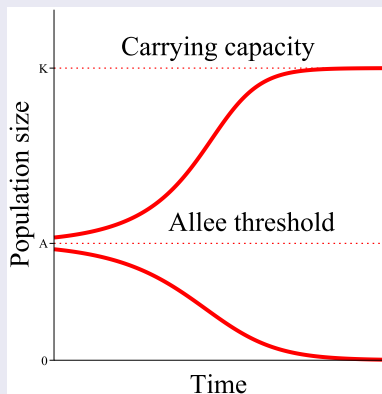
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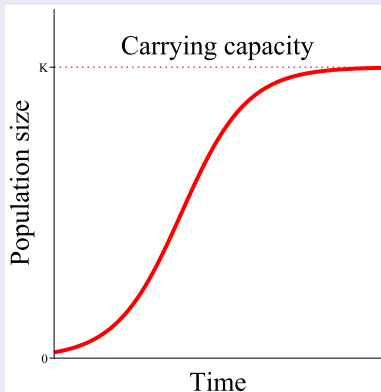
(Strong) Allee effect



$$\dot{N} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{A} - 1\right)$$

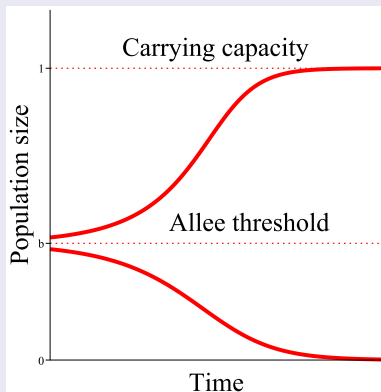
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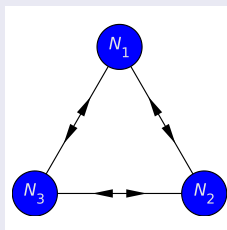


$$\dot{N} = N(1 - N)(N - b)$$

Main model

n -connected populations with Allee effect.

$$\dot{N}_i = N_i(1 - N_i)(N_i - b) - (n - 1)aN_i + \sum_{\substack{j=1 \\ j \neq i}}^n aN_j, \quad i = 1, \dots, n.$$



n -connected populations with Allee effect.

$$N_i(1 - N_i)(N_i - b) - (n - 1)aN_i + \sum_{\substack{j=1 \\ j \neq i}}^n aN_j = 0, \quad i = 1, \dots, n.$$

This parametric system has n variables, N_i s, and 2 parameters, the Allee effect parameter b and the strength of connectivity a .

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Questions

- What happens for not very small and not very large values of a ?
- Is it true that the number of non-negative solutions is non-increasing with respect to a ?

Approach no. 1

Original system

$$\{x^2 + bx + c = 0\}$$

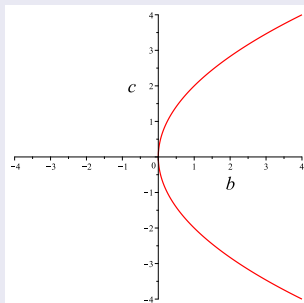
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⇓ Discriminant Variety (Elimination via Gröbner bases).

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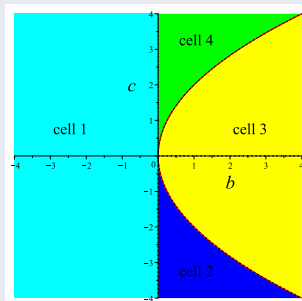
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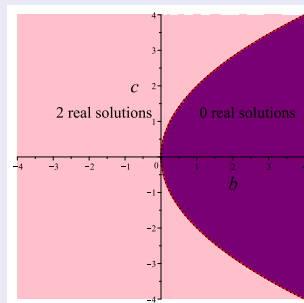
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⇓ (open) CAD

$$\#(f^{-1}(0) \cap \mathbb{R}) =$$

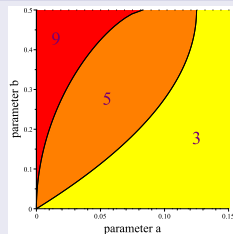
$$\begin{cases} 2 & ; (c, b) \in \text{cells } 1, 2, 4 \\ 0 & ; (c, b) \in \text{cell } 3 \end{cases}$$



Using approach 1 (used in ref. 3)

Remember the n -patches model.

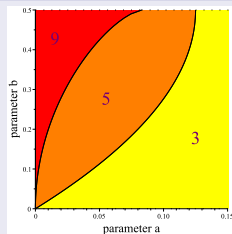
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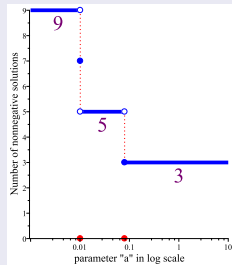
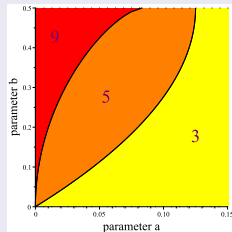
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By fixing the value of b , Maple* can compute the 1-dimensional CAD* of the model for $n = 2, 3, 4$, but not $n = 5$.

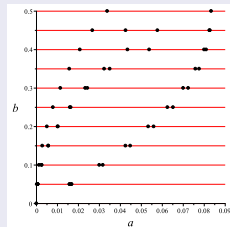
As an example the case $n = 2$ for $b = 0.2$ is shown here.



Approach no. 2 (introduced and used in ref. 2)

Using a numeric search.

Finding sections of the 2-dimensional CAD using 1-dimensional CADs for some finite samples of a parameter.

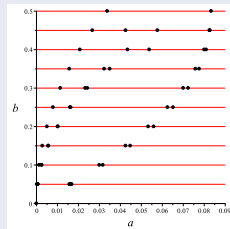


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Then using a numeric search to find where the behavior changes.

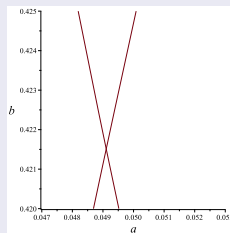
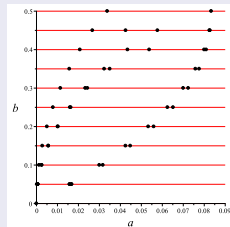
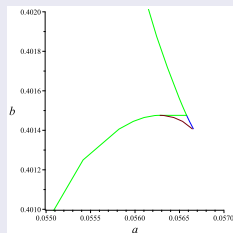


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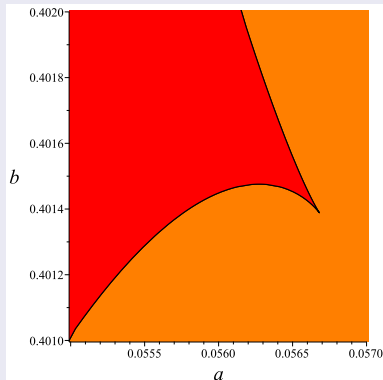
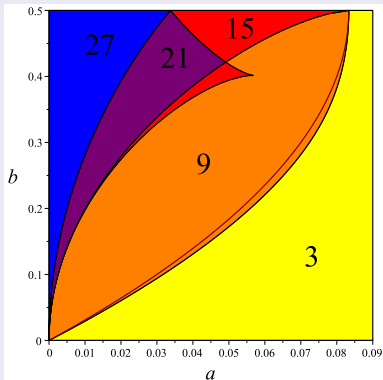
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Up to 7 digits accuracy after the decimal point

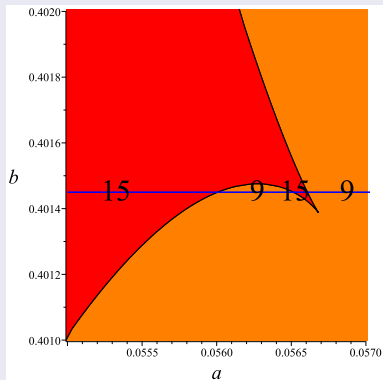
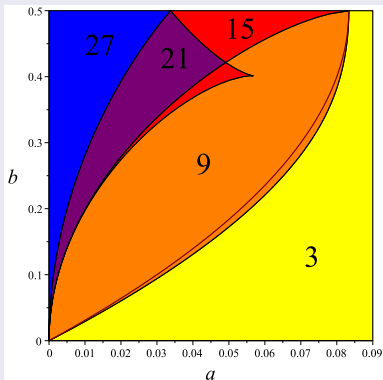


A discovery

The number of steady states is not always decreasing monotonically by increasing a .

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Some facts

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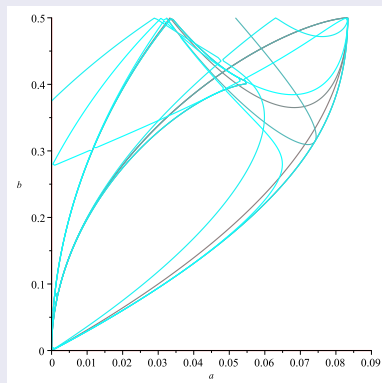
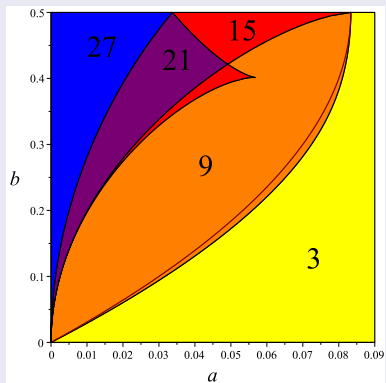
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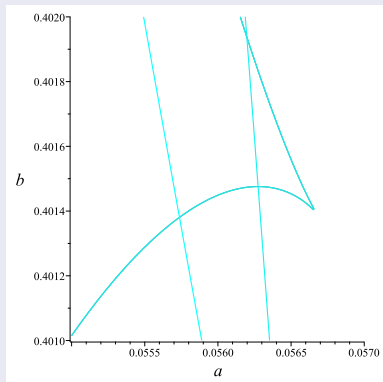
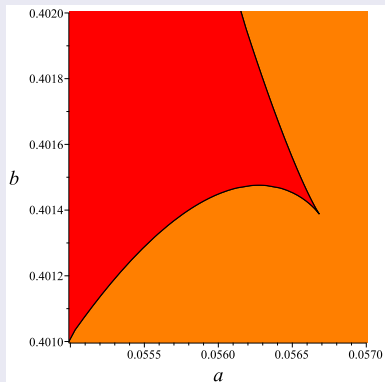
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- ② Using projection operator of CAD algorithm on the original system to eliminate variables gets one a superset of the discriminant variety.
- ③ Using Equational Constraints (ECs) the computations in the previous step becomes singly exponential.
- ④ An open CAD with respect to the new variety also answers the questions of our interest.

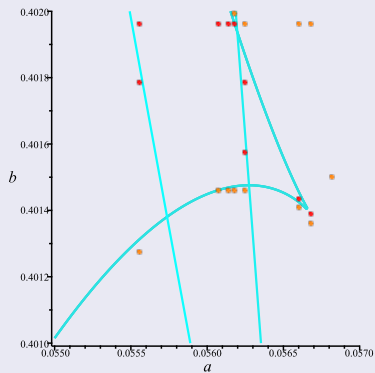
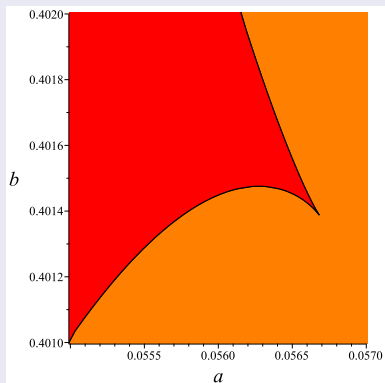
Comparison



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What is new?

We implemented a new algorithm in Maple that can handle larger size examples of parametric system of equations (and inequalities) with the following properties;

- ① It is free of Gröbner bases computation.
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Does it mean that the former numerical approach is not interesting anymore?

No, one still can equip approach 2 on top of approach 3 and go even further in the size of examples that can be handled by a normal computer.

References

- ① AmirHosein Sadeghimanesh, Matthew England, *Improving algebraic tools to study bifurcation sequences of population models*, in progress, 2021.
- ② Gergely Röst, AmirHosein Sadeghimanesh, *Exotic bifurcations in three connected populations with Allee effects*, bioArxiv, 2021, DOI: 10.1101/2021.02.03.429609.
- ③ Gergely Röst, AmirHosein Sadeghimanesh, *Unidirectional migration of populations with Allee effect*, bioArxiv, 2021, DOI: 10.1101/2021.06.24.449708.

Thank you for listening.