Generalised Hirota operators and bases of generalised elliptic functions

University of Glasgow

Matthew England

School of Mathematics and Statistics

Matthew.England@glasgow.ac.uk

www.maths.gla.ac.uk/∼mengland/

December 2011

1. Introduction and motivation

Elliptic functions have a single complex variable and two independent periods. They have been the subject of much study since their discovery and have been extensively used to enumerate solutions of non-linear wave equations. They occur in many physical applications; traditionally the arc-length of the lemniscate and the dynamics of spherical pendulums, but also in cryptography and soliton solutions to the KdV equation. Recent times have seen a revival of interest in the theory of their generalisations, the Abelian functions, which have multiple independent periods, or more accurately, are periodic with respect to a multi-dimensional period lattice. The lattice is usually defined in association with an underlying algebraic curve. These functions also have a wide range of applications. They not only give further solutions to the KdV equation, but also solutions to other integrable equations from the KP-hierarchy. They have also been used to construct reductions of the Benney hierarchy, to model the motion of a double pendulum and to describe geodesic motions in certain space-time metrics.

The most common definition of Abelian functions is as generalisations of the Weierstrass $\wp$-function. This has allowed for investigation of their properties, with recent progress following both from new theory, and advancement of efficient symbolic computation techniques. One key problem remaining is the construction of bases for the vector spaces of the functions. The Riemann-Roch theorem can be applied to determine the dimension of the spaces, while series expansions can be employed to check linear independence. The unresolved issue is identifying enough suitable functions of a given type. We describe various approaches to this problem and present a new method which can define infinite sequences of classes of Abelian functions of a given type. Key to this is the definition of generalised Hirota operators. The solution of the basis problem allows for properties of the Abelian functions to be derived, such as differential equations and addition formulae, and for their use in applications.

Weierstrass elliptic functions

We recall the Weierstrass functions which serve as a model for the generalisation.

Karl Weierstrass 1815-1897

An elliptic function is a meromorphic function $f(u), u \in \mathbb{C}$, which has two independent periods.

Weierstrass defined the elliptic $\wp$-function, which had poles occurring only when u is a sum of periods. It satisfied the following, where g_2, g_3 are constants dependent on the periods.

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3 \qquad (1)$$
$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2 \qquad (2)$$

The $\wp$-function and its derivative may be used to parametrise an elliptic curve and the $\wp$-function satisfies an addition formula representing the group addition law for points on such a curve.

Weierstrass also defined the elliptic σ-function by

$$\wp(u) = -\frac{d^2}{du^2}\ln[\sigma(u)] \qquad (3)$$

which has its own addition formula,

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v). \qquad (4)$$

2. Defining generalised elliptic functions

We define generalised elliptic functions or Abelian functions as those meromorphic functions periodic with respect to the standard period lattice of an associated algebraic curve of genus g. All the associated functions will be dependent on g variables, $\boldsymbol{u} = (u_1, u_2, ..., u_g)$. There is a well defined construction for (n,s)-curves, described for example in [6]. However, other curve models and constructions are a topic of investigation, for example in [1].

Given an (n,s)-curve we define the associated Kleinian σ-function as an entire function, satisfying a quasi-periodicity condition, (periodic up to an exponential factor), with leading order terms in its Maclaurin series given by a Schur-Weierstrass function. This function is given uniquely by a modified Riemann θ-function. We define Kleinian $\wp$-functions as log derivatives of $\sigma(\boldsymbol{u})$, in analogy to (3):

$$\wp_{i_1,i_2,...,i_m}(\boldsymbol{u}) = -\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\cdots\frac{\partial}{\partial u_{i_m}}\ln\sigma(\boldsymbol{u}),$$

where $i_1 \leq ... \leq i_m \in \{1,...,g\}$. We find these are all Abelian, with poles of order m when $\sigma(\boldsymbol{u}) = 0$. If we impose this notation on the elliptic case then we would denote $\wp \equiv \wp_{11}$, $\wp' \equiv \wp_{111}, \wp'' \equiv \wp_{1111}$ etc.

This approach was initiated by the work of Klein and Baker, with the general definitions pioneered by Buchstaber, Enolskii and Leykin in [2].

Felix Klein 1849 - 1925

H.F. Baker 1866 - 1956

References

[1] C. Athorne. Identities for hyperelliptic $\wp$-functions of genus one, two and three in covariant form. J. Phys. A: Math. & Theo. 41:415202, 2008.

[2] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin. Kleinian functions, hyperelliptic Jacobians and applications. Reviews in Math. and Math. Phys., 10:1125, 1997.

[3] J. C. Eilbeck, V. Z. Enolski, S. Matsutani, Y. Onishi, and E. Previato. Abelian functions for trigonal curves of genus three. Intl. Math. Research Notices, Vol. 2008:102139, 2008.

[4] M. England. Deriving bases for Abelian functions. Comput. Methods Funct. Theory, 11(2), 2011.

[5] M. England and J. C. Eilbeck. Abelian functions associated with a cyclic tetragonal curve of genus six. J. Phys. A: Math. & Theo. 42:095210 (27pp), 2009.

[6] A. Nakayashiki. On algebraic expressions of sigma functions for (n,s)-curves. Asian J. Math. 14(2):175212, 2010.

3. The basis problem

Denote by $\Gamma(m)$ the vector space of Abelian functions with poles of order at most m, defined upon the Jacobian of the curve with poles occurring only on the Θ-divisor; the points where the θ and σ-functions have their zeros and the Abelian functions their poles. We seek bases for these spaces.

The dimension of the space $\Gamma(m)$ is m^g by the Riemann-Roch theorem for Abelian varieties. The first step in constructing a basis is to include the entries in the preceding basis for $\Gamma(m-1)$. Then only functions with poles of order exactly m need to be sought. The m-index $\wp$-functions are natural candidates and are sufficient to solve the problem in the elliptic case, as described in Table 1.

Space	**Dim**	**Basis**
$\Gamma(0) = \Gamma(1)$	1	1
$\Gamma(2)$	2	$1, \wp$
$\Gamma(3)$	3	$1, \wp, \wp'$
⋮	⋮	⋮
$\Gamma(m)$	m	$1, \wp, \wp', \wp'', \dots, \wp^{(m-2)}$

Table 1: Table of bases for the elliptic case.

However, if $g > 1$ then further classes of functions are required to complete the bases. When $g = 2$ we see that an additional function is required in $\Gamma(3)$, with $\Xi = \wp_{11}\wp_{22} - \wp_{12}^2$ usually taken to fill this hole. While each term in Ξ has poles of order 4, these cancel to leave poles of order 3 overall.

Then, when considering subsequent spaces in the genus 2 case, the $\wp$-functions and the derivatives of Ξ are sufficient, as presented in Table 2. Here ∂_i indicates differentiation with respect to u_i and $\{\cdot\}$ all functions of that form. In this case the linear independence of the functions may be checked using a weight argument. These examples are exceptions as in general these sequences of spaces will not be finitely generated by differentiation.

Space	**Dim**	**Basis for $\Gamma(m)\backslash\Gamma(m-1)$**
$\Gamma(2)$	4	$1, \wp_{11}, \wp_{12}, \wp_{22}$
$\Gamma(3)$	9	$\wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}, \Xi$
$\Gamma(4)$	16	$\wp_{1111}, \dots, \wp_{2222}, \partial_1\Xi, \partial_2\Xi$
⋮	⋮	⋮
$\Gamma(m)$	m^2	$\{\wp_{i_1 \dots i_m}\}, \{\partial_{i_1}\dots\partial_{i_{m-2}}\Xi\}$

Table 2: Table of bases for the genus two case.

For example, in the hyperelliptic genus 3 case we can show that at least one new function is required at each stage in addition to derivatives of the previous basis. This is because the basis weight range is fixed with the maximal weight entry increasing with m by one more than the highest weight derivative.

Recently, the basis problem has been solved in several specific cases by defining new classes as algebraic combinations of $\wp$-functions chosen for poles cancelation. For example,

$$\mathcal{B}_{ijklm} = \wp_{ij}\wp_{klm} + \tfrac{1}{3}\big(\wp_{jk}\wp_{ilm} + \wp_{jl}\wp_{ikm} + \wp_{jm}\wp_{ikl} - 2\wp_{kl}\wp_{ijm} - 2\wp_{km}\wp_{ijl} - 2\wp_{lm}\wp_{ijk}\big)$$

always belongs to $\Gamma(3)$. In [4] such classes are presented in a systematic way and used to find new addition formulae and differential equations. While such an approach can be useful in specific cases and applications, it is not clear how it may be generalised and so we now take a different approach.

4. Generalised Hirota operators

Recall Hirota's bilinear operator defined as $\mathcal{D}_i = \partial/\partial u_i - \partial/\partial v_i$. We define the m-index Q-functions

$$Q_{i_1,i_2,...,i_m}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2}\mathcal{D}_{i_1}\mathcal{D}_{i_2}...\mathcal{D}_{i_m}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\Big|_{\boldsymbol{v}=\boldsymbol{u}}, \qquad i_1 \leq ... \leq i_n \in \{1,\dots,g\}, \qquad m \text{ even.}$$

They were first introduced in generality in [5], with the $m = 4$ case first appearing in [3] and that in turn generalising the original Q-function of Baker. Comparing the definitions we see that $Q_{ij} = \wp_{ij}$. In fact, the Q-functions are all Abelian with poles of order two and are sufficient to solve the basis problem in for $\Gamma(2)$. The Q-functions have motivated the following definitions.

Let $\boldsymbol{u^1}, \boldsymbol{u^2}, \dots, \boldsymbol{u^m} \in \mathbb{C}^g$ with $\boldsymbol{u^j} = (u_1^j, u_2^j, \dots, u_g^j)$. Define generalised Hirota operators by

$$\mathcal{H}_i^m = \sum_{j=1}^{m} \zeta^{j-1}\frac{\partial}{\partial u_i^j}, \qquad \text{where } \zeta \text{ is a primitive } m\text{th root of unity.}$$

Then define n-index, mth order $\mathcal{R}$-functions by

$$\mathcal{R}^m_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = \frac{(-1)}{m\sigma(\boldsymbol{u})^m}\mathcal{H}^m_{i_1}\mathcal{H}^m_{i_2}...\mathcal{H}^m_{i_n}\sigma(\boldsymbol{u^1})\sigma(\boldsymbol{u^2})\dots\sigma(\boldsymbol{u^m})\Big|_{\boldsymbol{u^1}=\boldsymbol{u^2}=\cdots=\boldsymbol{u}},$$

where $i_1 \leq ... \leq i_n \in \{1,\dots,g\}$ and n has m as a factor. By making specific choices of m, n we find

$$\mathcal{R}^m_{i_1,i_2,\dots,i_m}(\boldsymbol{u}) = \wp_{i_1,i_2,\dots,i_m}(\boldsymbol{u}) \qquad \text{and} \qquad \mathcal{R}^2_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = Q_{i_1,i_2,\dots,i_n}(\boldsymbol{u}).$$

So the $\wp$ and Q-functions are all subclasses of $\mathcal{R}$-functions. We find that the mth order $\mathcal{R}$-functions are Abelian with poles of order m and so can be used in the basis for $\Gamma(m)$. In the hyperelliptic genus 3 case, discussed above, we can now identify the maximal weight function in each basis $\Gamma(m)$ as the $2m$-index $\mathcal{R}$-function $\mathcal{R}^m_{2,2,\dots,2}(\boldsymbol{u})$. Further structure of the general bases is now under investigation.

Computation with series expansions

To check linear independence we may use the series expansions. The construction of these is an important tool, [5], but the computations involved can be large.

Computations may be performed exactly in a symbolic package, but code must be written to utilize simplifications in the theory. E.g:

- There are a set of Sato weights which render the theory of Abelian functions homogeneous, (see e.g. [5]). Taking this weight structure into account can reduce many calculations.
- All $\wp$-functions have a definite parity in $\boldsymbol{u}$ given by the number of their indices.

The expansions can also be employed, along with the bases, to derive sets of differential equations that generalise (1) and (2), as well as various addition formulae that generalise (4). See for example [3], [4], [5].