# SC-Square: Past Successes and Future Progress with Machine Learning

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**The 6th SC-Square Workshop (SC$^2$ 2021)**
SIAM Conference on Applied Algebraic Geometry (AG21)
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# Outline

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# SC-Square Definition



**SC**: **S**atisfiability **C**hecking Community
Interested in algorithms to check satisfiability of logic problems with variables from a variety of mathematical domains. Implement solutions in SAT/SMT solvers.

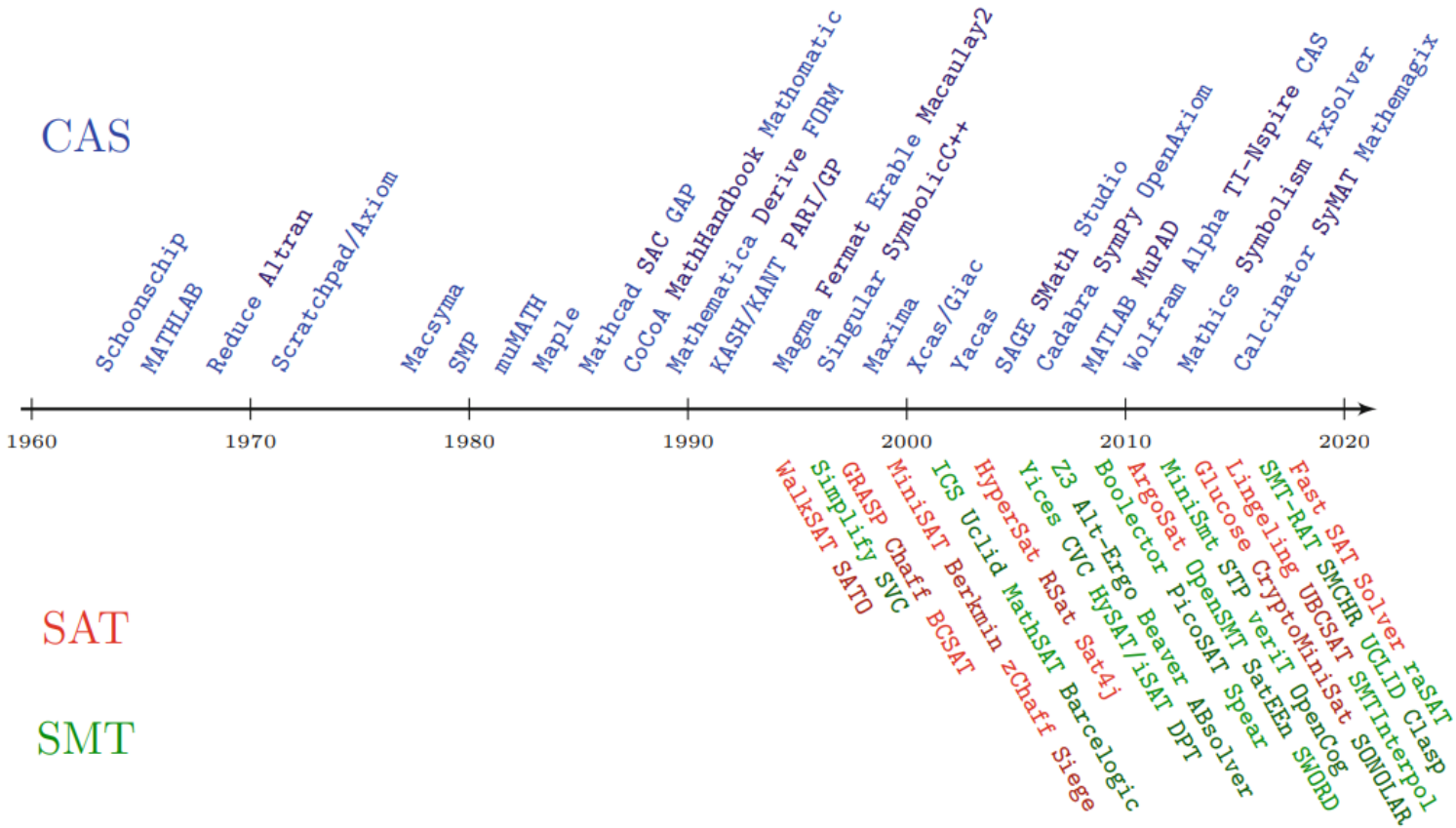
**SC**: **S**ymbolic **C**omputation Community
Interested in algorithms to perform algebraic computations, such as polynomial computations over real and complex numbers. Implement solutions in Computer Algebra Systems.

**SC$^2$**: SC-Square Community
Interested in both SCs above!

# Separate Histories



Until the last decade the two communities rarely interacted.




Timeline by E. Ábrahám Ábrahám [2015]

# Why Interact?



**Satisfiability Modulo Theories** (SMT) refers to the SAT decision problem where the atoms need not be Boolean variables but can instead be predicates over variables from other domains (e.g. arithmetics, data structures, uninterpreted functions).

A common framework for solving such problems separates out the reasoning in the logic (performed by a SAT solver) from the reasoning in the theory. The theory reasoning is done with algorithms / software from the domain of interest. For many theories the domain of interest is traditionally studied as Symbolic Computation.

In the opposite direction; the integration of SAT solvers into computer algebra systems can allow these tools to offer more powerful logical reasoning.

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# EU Project



The EU funded **The SC-Square Project** from 2016-2018 [Ábrahám et al., 2016]. The aim was to bridge the gap between the communities to produce individuals who can combine the knowledge and techniques of both fields to resolve problems currently beyond the scope of either.

The project funded new collaborations, new tool integrations, proposals on extensions to the SMT-LIB language standards, new collections of benchmarks, two summer schools (in 2017 and 2018) and this SC-Square Workshop Series.

Although the project finished, the workshop series continues, and the website and mailing lists remain active. www.sc-square.org

Volume 100 of the Journal of Symbolic Computation was an SC-Square themed special issue [Davenport et al., 2020].

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# SC-Square Workshop Series (Slide 5/19)

2016 Timisoara, Romainia (as part of SYNASC 2016).

2017 Kaiserslautern, Germany (alongside ISSAC 2017).

2018 Oxford, UK (as part of FLoC 2018).

2019 Bern, Switzlerland (as part of SIAM AG19)

2020 Paris, France (online) (alongside IJCAR 2020)

2021 Texas, USA (online) (as part of SIAM AG21)

Take place as part of / alongside established conferences (alternate between computational algebra and logic).

Each year there are two chairs, one from each SC.

Now as many conferences outside the EU project as inside.
Funding obtained to support workshops in 2022-23.

# Non-linear real arithmetic I



Algorithms for working with systems of non-linear polynomials are a core topic for symbolic computation but also key for SMT in the [QF_]NRA logic. This area has seen a lot of work in SC$^2$:

NLSAT Algorithm of Jovanovic and de Moura [2012] re-purposed the theory of cylindrical algebraic decomposition by Collins [1975] to use in the MCSAT proof framework [de Moura and Jovanović, 2013].

The SMT-RAT Toolbox has implementations of a variety of computer algebra tools for use in SMT [Kremer and Ábrahám, 2018].

The algorithms must be adapted for the purpose (incremental by constraint and produce conflicting cores). E.g. Cylindrical Algebraic Decomposition [Kremer and Ábrahám, 2020], Groebner Bases [Junges et al., 2013].

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# Non-linear real arithmetic II



Conflict Driven Cylindrical Algebraic Coverings of Ábrahám et al. [2021] repurpose CAD technology like NLSAT.

Like NLSAT they produce coverings instead of a decomposition but unlike NLSAT this is a single algebraic procedure which may be used in a DPLL(T) framework.

The NuCAD algorithm of Brown [2015] is also inspired by NLSAT to construct one cell by cell using local information.

Can be used also for Quantifier Elimination [Brown, 2017].

Redlog+VeriT combination of computer algebra with heuristics based on interval constraint propagation and subtropical satisfiability [Fontaine et al., 2018].

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# Other SC-Square Successes I



**Maple** can now read to and from SMT-LIB [Forrest, 2017] and ships with the both the Z3 and MapleSAT.

**CoCoALib** C++ Library that underpins CoCoA [Abbott and Bigatti, 2014] used by MathSAT and SMT-RAT.

**MathCheck** has made significant progress on a variety of combinatorics problems from a combination of SAT-solvers and computer algebra, enumerating new cases and verifying conjectures. E.g. Williamson Matrices [Bright et al., 2020]; Golay Pairs [Bright et al., 2018]; Good Matrices [Bright et al., 2019].

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# Other SC-Square Successes II



**Boolean SAT** has also benefited from symbolic computation via Boolean Groebner Bases and parallel computation on both the conjunctive and algebraic normal forms of a problem [Horáček and Kreuzer, 2020].

**Circuit Verification** Algebraic techniques key for circuit verification [Kaufmann, 2021], and in combination with SAT-solvers [Kaufmann et al., 2019].

**Proofs** work on natural style proofs [Jebelean, 2018], [Ábrahám et al., 2020].

**New Applications** in Economics [Mulligan et al., 2018], Dynamic Geometry [Vajda and Kovács, 2020], knot theory [Lisitsa and Vernitski, 2017, Meesum and Prathamesh, 2018].

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# ML for SC$^2$ Technology? (Slide 10/19)

**Machine Learning** tools use statistics and large quantities of data to learn how to perform tasks not explicitly programmed.
They offer only probabilistic guidance, but both SC communities produce software that prizes exact results. Is there interest for SC$^2$?

**Yes**. ML techniques can make non-critical choices and guide searches. Used this way, the ML performance has no affect on the mathematical correctness of the final result, but can make a substantial contribution to both computational efficiency and presentation of the end result.

Often, ML can make such choices better than the user, developer, or currently employed human-made heuristics.

Note that this approach differs from say the Facebook Research work of Lample and Charton (2020) which used ML to predict the end result of a mathematical process directly.

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# Examples from the literature (Slide 11/19)

- Xu et al. [2008] won SAT competitions with their portfolio solver which selected the best SAT solver for the instance.
- Liang et al. [2016] used a machine learnt branching heuristic in their MapleSAT solver for formulae in Boolean logic.
- Wu [2017] used a regression model to predict the satisfiability of formulae after fixing some variables; to select initial values.

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- Kuipers et al. [2015] used a Monte-Carlo tree search to find the representation of polynomials that are most efficient to evaluate in their CAS.
- Huang et al. [2016] used a support vector machine to decide whether or not to precondition CAD with Gröbner Bases.
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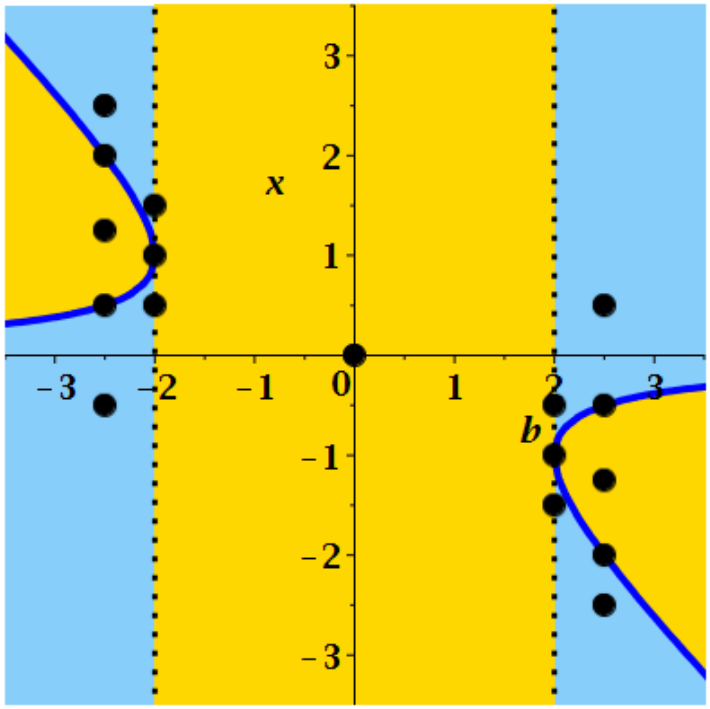
# CAD and QE



A CAD is a decomposition of $\mathbb{R}^n$ that may be algorithmically produced from a set of polynomials such that each polynomial has invariant sign on each cell. E.g. CAD below for $\{x^2 + bx + 1\}$. The invariance means we can answer questions on polynomials by testing a finite number of sample points.



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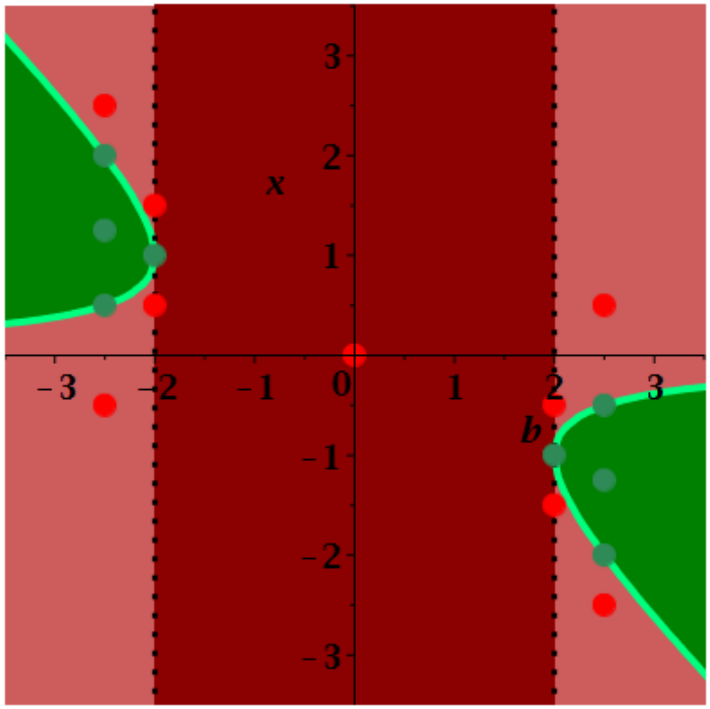
A key application is quantifier elimination. E.g.

$$\exists x, x^2 + bx + 1 \leq 0$$

Tag cells in the CAD true or false according to polynomial sign.



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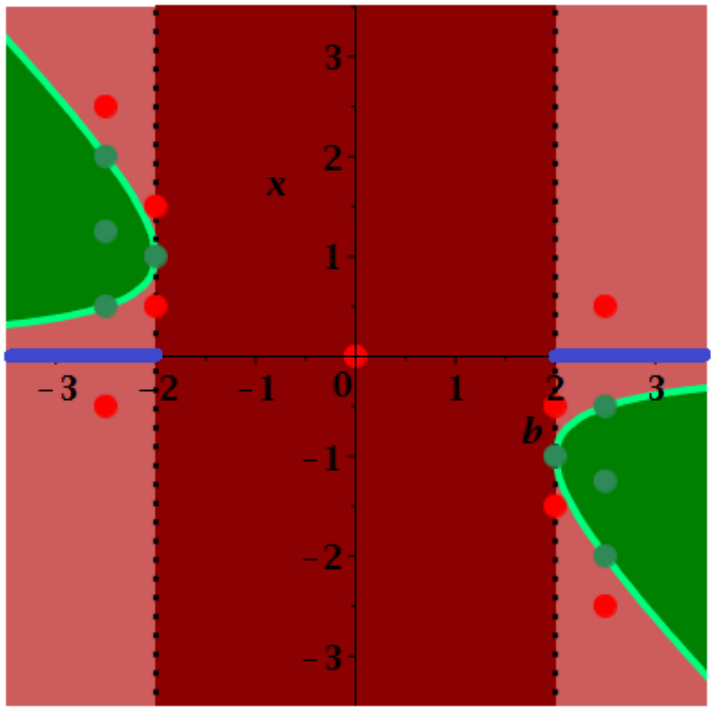
Tag cells in the CAD true or false according to polynomial sign. Then project true cells and take disjunction. $\implies b \leq -2 \vee b \geq 2$.



# CAD Variable Ordering Choice



CAD has doubly exponential complexity in the number of variables! The complexity is felt in practice. But careful optimisations and pre-processing can *push back the doubly exponential wall* and bring new applications in scope.

A particularly important optimisation is the variable ordering. For QE one must order variables as they are quantified; but there is no restriction on free variables and adjacent quantifiers of the same type may be swapped.

The ordering is well known to dramatically affect both the number of cells and the time to compute them.

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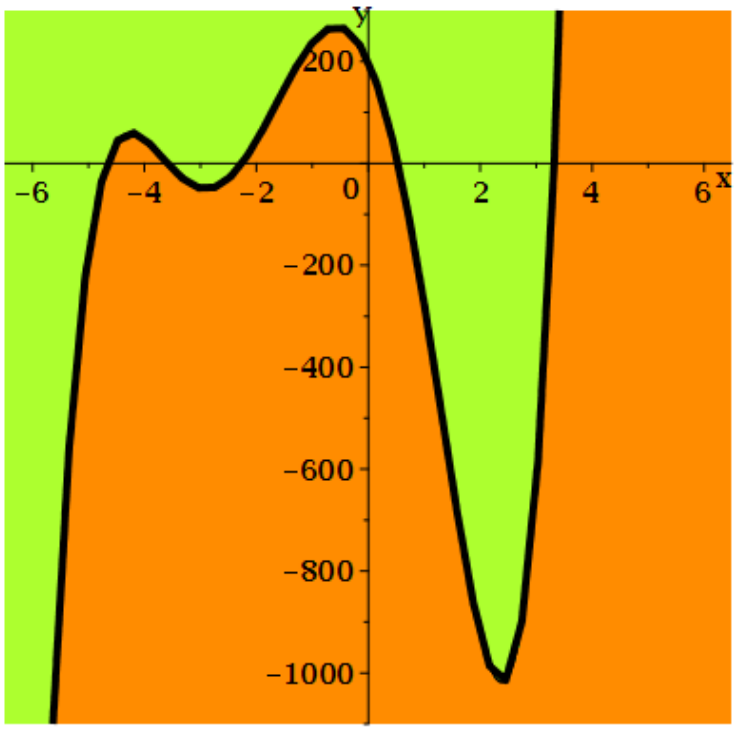
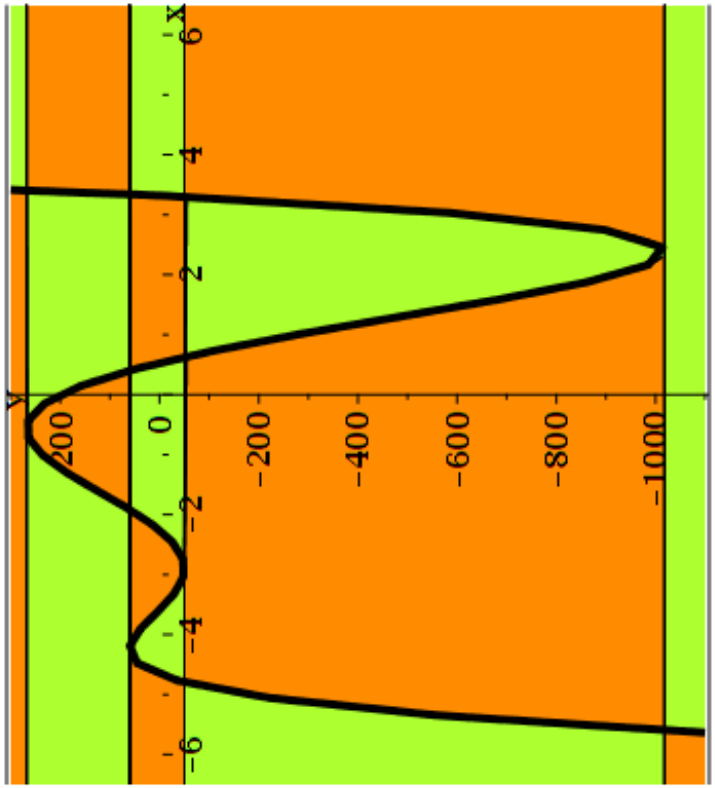
# CAD Variable Ordering Example



CAD for polynomial $y - 3x^5 + 20x^4 - 10x^3 - 240x^2 - 250x + 200$.
With $y \succ x$ a sign-invariant CAD has 3 cells, with $y \prec x$ it is 59.



# Prior Work on CAD Variable Ordering (Slide 15/19)

There have been a succession of human designed heuristics: [Brown, 2004], [Dolzmann et al., 2004], [Bradford et al., 2013], [Wilson et al., 2014]. They are increasingly expensive and have variable performance: none of them are perfect.

Huang et al. [2014] used a support vector machine to pick which of the above heuristics to follow: this meta-heuristic did better than any one heuristic individually. The experiments also identified substantial subsets on which each human-designed heuristic was superior to the others.

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# Our Recent Work I



EPSRC Project EP/R019622/1 (2018-2020) *Embedding Machine Learning within Quantifier Elimination Procedures* further investigated the use of ML for CAD variable ordering choice.

- Experimented with several ML classifiers in `sklearn`: they all do better than the existing human-made heuristics [England and Florescu, 2019].
- New approach to generate features of the input polynomials [Florescu and England, 2019].

ML classifiers are trained not on input polynomials themselves but vectors of floating point numbers derived from them (the **features**). Prior work used features inspired by Brown's heuristic. The new approach enumerated all (appropriate) combinations of basic (i.e. cheap) functions used by Brown and removes redundancies to leaves 78 features in 3-variable and 105 in 4.

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# Our Recent Work II



- Proposed a more suitable measure of ML classifier accuracy and used this to write an improved method for cross-validation hyper-parameter selection in `sklearn` [Florescu and England, 2020a].

Classifiers usually seek the optimal (ordering which used least resources) but this gives no difference between *almost optimal* and *terrible*. Instead define accurate if close to the optimal and minimise directly on computation time during cross-validation.

- Combined into a software pipeline [Florescu and England, 2020b] and released for free as a Zenodo repository: https://doi.org/10.5281/zenodo.3731703

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# Successes and Challenges



Experiments showed that the new classifiers tried did better than SVM; that all classifiers benefited from the additional features; that all classifiers benefited from the new cross validation procedure (with and without the additional features).

On our 3-variable dataset: human made heuristics achieved times 27% above the minimum; ML achieved times 6% above.

On our 4-variable dataset: human made heuristics achieved times 98% above the minimum; ML achieved times 67% above.

**Challenges:**

- Dealing with exponential growth in the number of choices.
- Finding datasets that are sufficiently varied, and representative of all use-cases.
- Finding datasets large enough for deep learning.

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# Contact Details and Advert



## Contact Details

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**Fully Funded PhD Position** available at Coventry to work on Machine Learning to Improve Symbolic Integration and Symbolic Simplification. Sponsored by Maplesoft.
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**Deadline to Apply:** 13th September 2021
**Interviews and Decision:** End September
**PhD Start:** Jan 2022

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