# Machine Learning for Reduce/Redlog?

## Some Ideas and Numbers for Discussion

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ICMS 2018, Notre Dame, IN, July 26, 2018

# Redlog and Reduce

Our computer logic system Redlog is technically a package of the CAS Reduce.

`http://www.redlog.eu`

- interactive system, quantifier elimination and decision for many domains, normal forms, simplification, construction and decomposition of large formulas, ...
- interfaces to Qepcad B, Gurobi, Mathematica, Z3, ...
- 400+ citations, mostly with applications, in the literature:

  geometry, planning, scientific computation, verification, physics and engineering, chemistry, life sciences, ...

- Redlog development since 1992 as part of the CAS Reduce [Hearn, 1968]
- Reduce/Redlog open-source (free-BSD) on Sourceforge since 12/2008

  `http://reduce-algebra.sourceforge.net`

- 80,000+ downloads since 2008 (12,000+ in the last 365 days)
- 500+ SVN commits per year

# Real Quantifier Elimination (QE)

Let $p \doteq x^2 + xy + b$ and $q \doteq x + ay^2 + b \in \mathbb{Z}[a, b, x, y]$

$$\varphi \doteq \forall x \exists y (p > 0 \land q \leq 0)$$

$\varphi$ formally asks for a necessary and sufficient condition in terms of parameters $a$, $b$.
[One possible] solution is

$$a < 0 \land b > 0$$

**Syntax:** The language of ordered rings specifies admissible symbols:

$$L = \{0, 1, +, -, \cdot, =, \leq, <, \geq, >, \neq\}.$$

**Semantics:** All functions, and relations have their usual interpretation over $\mathbb{R}$.

QE is a generalization of Satisfiability Modulo Theories Solving (SMT) for real arithmetic (QF_LRA, LRA, QF_NRA, NRA).

# $SC^2$: Symbolic Computation + Satisfiability Checking

EU Horizon 2020 CSA Project (2016–2018)

| | | |
|---|---|---|
| University of Bath | UK | James Davenport et al. |
| RWTH Aachen | Germany | Erika Ábrahám |
| Fondazione Bruno Kessler | Italy | Alberto Griggio et al. |
| Università degli Studi di Genova | Italy | Anna Bigatti |
| Maplesoft Europe Ltd. | Germany | Jürgen Gerhard et al. |
| Université de Lorraine (LORIA) | France | Pascal Fontaine |
| Coventry University | UK | Matthew England |
| University of Oxford | UK | Martin Brain |
| Universität Kassel | Germany | Werner Seiler et al. |
| Max-Planck-Institut für Informatik | Germany | Thomas Sturm |
| Universität Linz | Auatria | Bruno Buchberger et al. |

# The Virtual Substitution Method for the Reals

**Given:** $\varphi \doteq Q_k \dots Q_1 \psi$

**Principal Strategy:** prenex formula, $\forall x_1 \psi \longleftrightarrow \neg \exists x_1 \neg \psi$,

$$\exists x_n \dots \exists x_2 \, \exists x_1 \psi \quad \longleftrightarrow \quad \exists x_n \dots \exists x_2 \; \mathtt{simplify} \left( \bigvee_{(\gamma, t) \in E} \gamma \wedge \psi[x_1 /\!/ t] \right).$$

$E = \{\dots, (\gamma, t), \dots\}$ is a finite elimination set, e.g.,

$$a x_1^2 - 3x_1 + 7 \leq 0 \;\rightsquigarrow\; \left( a \neq 0 \wedge (-3)^2 - 4 \cdot a \cdot 7 \geq 0, \frac{3 + \sqrt{(-3)^2 - 4 \cdot a \cdot 7}}{2 \cdot a} \right) \in E.$$

Also substitute $\pm\infty$, nonstandard $t \pm \varepsilon$ with $<$, abstract roots for higher degrees.

$[x_1 /\!/ t]$ : atomic formulas $\rightarrow$ quantifier-free formulas

**!** Alternatively, we could guard every atomic formula in $\psi$ with $\gamma \wedge \dots$; simpler Boolean structure vs. fewer atomic formulas.

# An Example for a Virtual Substitution

**Formal quadratic solution of $ax_1^2 + bx_1 + c = 0$ into equation $\alpha x_1 + \beta = 0$:**

Formal substitution for intuition:

$$(\alpha x_1 + \beta = 0)[x_1 / \tfrac{-b+\sqrt{\Delta}}{2a})] \doteq \alpha \frac{-b + \sqrt{\Delta}}{2a} + \beta \doteq \frac{(-\alpha b + 2\beta a) + \alpha\sqrt{\Delta}}{2a}$$

Virtual substitution:

$$(\alpha x_1 + \beta = 0)\left[x_1 /\!\!/ \left(\tfrac{-b+\sqrt{\Delta}}{2a}\right)\right] \doteq (-\alpha b + 2\beta a)^2 = \alpha^2\Delta \wedge (-\alpha b + 2\beta a)\alpha \leq 0$$

! Increase in degrees is bad for efficiency of the method.

Factorization is an option, but complicated boolean structure is also bad.

$$fg > 0 \longleftrightarrow (f > 0 \wedge g > 0) \vee (f < 0 \wedge g < 0)$$

! Euclidean Algorithm vs. Exended Zassenhaus GCD in simplification

# Hopf Bifurcations in System Biology and Chemistry

**Boulier et al. Proc. Algebraic Biology 2007**

*On proving the absence of oscillations in models of genetic circuits*

Discusses circadian clock within a unicellular green alga.

Principle problem (theoretically very difficult):

Given: a system of parametric ordinary differential equations (with a polynomial vector field) built using mass action kinetics:

$$
\begin{aligned}
\dot{G}(t) &= \vartheta \cdot \left(\gamma_0 - G(t) - G(t)P(t)^n\right) \\
\dot{M}(t) &= \lambda G(t) + \gamma_0 \mu - M(t) \\
\dot{P}(t) &= n\alpha \cdot \left(\gamma_0 - G(t) - G(t)P(t)^n\right) + \delta \cdot \left(M(t) - P(t)\right)
\end{aligned}
$$

Question: Are there (biologically meaningful) ranges of values for the parameters and variables, where **oscillating trajectories** or **limit cycles** can be found?

# Hopf Bifurcations in System Biology and Chemistry

Related but easier problem:

Modified question: Are there (biologically meaningful) ranges of values for the parameters and variables that give rise to a **Hopf Bifurcation** (i.e., a pair of eigenvalues of the Jacobian of the vector field gets purely imaginary, and all other eigenvalues have negative real parts.)

- This can be straightforwardly translated into real algebra [Liu 1994], [El Kahoui–Weber 2000].
- For fixed $n \in \mathbb{N}$, $n \geq 2$, we obtain a first-order formula $\varphi_n$.
- $\varphi_n$ holds for some choice of parameters iff a Hopf bifurcation exists.
- $\exists\varphi_n$ holds iff there is such a choice of parameters at all (easier!)

! More quantifiers are easier due to more flexibility with heuristic variable selection strategies (order of elimination of $\exists$ quantifiers).

# Hopf Bifurcations in System Biology and Chemistry

$$\varphi_9 = \exists v_2 \exists v_1 \exists v_3 (\text{all variables except } \lambda \text{ are strictly positive} \wedge$$
$$0 = \vartheta(\gamma_0 - v_1 - v_1 {v_3}^9) \wedge$$
$$0 = \lambda v_1 + \gamma_0 \mu - v_2 \wedge$$
$$0 = 9\alpha(\gamma_0 - v_1 - v_1 {v_3}^9) + \delta(v_2 - v_3) \quad \wedge \quad H_{n-1} = 0 \wedge H_{n-2} > 0)$$

$$H_{n-1} = 162\vartheta {v_3}^{17}\alpha v_1 + 162\vartheta\alpha v_1 {v_3}^8 + 162\alpha v_1 {v_3}^8\delta + \vartheta + 2\vartheta {v_3}^9\delta + \vartheta^2 {v_3}^{18}\delta + \vartheta {v_3}^9\vartheta\delta + 81\alpha v_1 {v_3}^8\vartheta\delta + 81\alpha v_1 {v_3}^{17}\vartheta\delta + \delta^2 + \vartheta\delta^2 + \vartheta^2\delta + \vartheta^2 + 2\vartheta^2 {v_3}^9 + \vartheta^2 {v_3}^{18} + 6561\alpha^2 {v_1}^2 {v_3}^{16} + 2\vartheta^2 {v_3}^9\delta + \delta + 81\alpha v_1 {v_3}^8 + \vartheta {v_3}^9\delta^2 - 9\lambda\vartheta v_1 {v_3}^8\delta$$

$$H_{n-2} = \vartheta\delta + \vartheta {v_3}^9\delta + 9\lambda\vartheta v_1 {v_3}^8\delta$$

# After Elimination of (Only) $\exists\alpha$ and $\exists\vartheta$

$p_{26} \neq 0 \land p_{25} \neq 0 \land p_{24} \neq 0 \land p_{23} \neq 0 \land p_{20} \neq 0 \land p_{12} \neq 0 \land p_{11} p_{17} p_{18} p_{20} \geq 0 \land (p_{11} \neq 0 \land (p_{26} \geq 0 \land p_{22} >$
$0 \land p_{20} \geq 0 \land p_{19} > 0 \land p_{15} > 0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} \geq 0 \land p_{12} p_{16} < 0 \land p_{11} p_8 p_9 < 0 \land$
$p_{12} p_{14} p_{20} p_{23} p_{24} p_{25} p_4 \geq 0 \land p_1 = 0 \land (p_{17} \neq 0 \lor p_{11} p_{12} p_{17} < 0) \land (p_{12} p_{20} p_{23} p_{24} p_{25} p_7 < 0 \lor p_2 < 0) \land (p_{17} \neq$
$0 \land p_{16} p_{17} < 0 \lor p_{16} p_{17} < 0 \land p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} \leq 0) \lor p_{26} \geq 0 \land p_{22} > 0 \land p_{20} \geq 0 \land p_{19} > 0 \land p_{15} >$
$0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} \geq 0 \land p_{12} p_{16} < 0 \land p_{11} p_8 p_9 < 0 \land p_{12} p_{14} p_{20} p_{23} p_{24} p_{25} p_4 \geq 0 \land p_1 = 0 \land (p_{17} \neq 0 \lor$
$p_{11} p_{12} p_{17} > 0) \land (p_{12} p_{20} p_{23} p_{24} p_{25} p_7 < 0 \lor p_2 < 0) \land (p_{17} \neq 0 \land p_{16} p_{17} < 0 \lor p_{16} p_{17} < 0 \land$
$p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} \leq 0)) \lor p_{11} \neq 0 \land (p_{26} \geq 0 \land p_{22} > 0 \land p_{20} \geq 0 \land p_{19} > 0 \land p_{15} > 0 \land p_{14} = 0 \land$
$p_{13} = 0 \land p_{12} \geq 0 \land p_{12} p_{20} p_{23} p_{24} p_{25} p_7 < 0 \land p_{12} p_{14} p_{20} p_{23} p_{24} p_{25} p_4 \leq 0 \land p_2 > 0 \land p_1 = 0 \land (p_{17} \neq 0 \lor$
$p_{11} p_{12} p_{17} < 0) \land (p_{12} p_{16} < 0 \lor p_{11} p_8 p_9 > 0) \land (p_{17} \neq 0 \land p_{16} p_{17} < 0 \land p_{11} p_8 p_9 < 0 \lor$
$p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} \geq 0 \land (p_{16} p_{17} < 0 \lor p_{17} \neq 0 \land p_{11} p_8 p_9 > 0)) \lor p_{26} \geq 0 \land p_{22} > 0 \land p_{20} \geq 0 \land p_{19} >$
$0 \land p_{15} > 0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} \geq 0 \land p_{12} p_{20} p_{23} p_{24} p_{25} p_7 < 0 \land p_{12} p_{14} p_{20} p_{23} p_{24} p_{25} p_4 \leq 0 \land p_2 > 0 \land$
$p_1 = 0 \land (p_{17} \neq 0 \lor p_{11} p_{12} p_{17} > 0) \land (p_{12} p_{16} < 0 \lor p_{11} p_8 p_9 > 0) \land (p_{17} \neq 0 \land p_{16} p_{17} < 0 \land p_{11} p_8 p_9 < 0 \lor$
$p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} \geq 0 \land (p_{16} p_{17} < 0 \lor p_{17} \neq 0 \land p_{11} p_8 p_9 > 0)))) \lor p_{26} \neq 0 \land p_{25} \neq 0 \land p_{24} \neq 0 \land p_{23} \neq$
$0 \land p_{20} \neq 0 \land p_{11} \neq 0 \land p_{10} \neq 0 \land (p_{11} p_8 p_9 < 0 \lor (p_{16} = 0 \lor p_{12} = 0 \lor p_{10} p_{11} p_{12} p_{16} p_{20} p_{23} p_{24} p_{25} > 0) \land (p_9 =$
$0 \lor p_8 = 0)) \land (p_{26} \geq 0 \land p_{22} > 0 \land p_{21} = 0 \land p_{20} \geq 0 \land p_{19} > 0 \land p_{15} > 0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} > 0 \land$
$p_{10} p_{20} p_{23} p_{24} p_{25} p_6 < 0 \land p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} > 0 \land (p_{17} \neq 0 \lor p_{11} p_{12} p_{17} < 0) \lor p_{26} \geq 0 \land p_{22} > 0 \land p_{21} =$
$0 \land p_{20} \geq 0 \land p_{19} > 0 \land p_{15} > 0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} > 0 \land p_{10} p_{20} p_{23} p_{24} p_{25} p_6 < 0 \land$
$p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} > 0 \land (p_{17} \neq 0 \lor p_{11} p_{12} p_{17} > 0)) \lor p_{26} > 0 \land p_{24} \neq 0 \land p_{23} \neq 0 \land p_{22} > 0 \land p_{21} =$
$0 \land p_{20} > 0 \land p_{19} > 0 \land p_{17} \neq 0 \land p_{15} > 0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} > 0 \land p_{10} p_{20} p_{23} p_{24} p_{25} p_6 < 0 \land$
$p_{10} p_{11} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} > 0 \land (p_{11} p_8 p_9 < 0 \lor (p_{16} = 0 \lor p_{10} p_{11} p_{12} p_{16} p_{20} p_{23} p_{24} p_{25} > 0) \land (p_9 = 0 \lor p_8 = 0)) \lor$
$p_{26} > 0 \land p_{22} > 0 \land p_{21} = 0 \land p_{20} > 0 \land p_{19} > 0 \land p_{15} > 0 \land p_{14} = 0 \land p_{13} = 0 \land p_{12} > 0 \land (p_{24} = 0 \lor p_{23} =$
$0) \land (p_6 \neq 0 \lor p_{24} \neq 0 \land p_{23} \neq 0) \land (p_5 < 0 \lor p_5 = 0 \land (p_{20} p_3 < 0 \lor p_3 = 0 \land p_{11} p_{12} p_6 < 0)) \land$
$(p_{10} p_{12} p_{17} p_{20} p_{23} p_{24} p_{25} < 0 \lor (p_{24} = 0 \lor p_{23} = 0 \lor p_{17} = 0) \land (p_{12} p_{17} p_5 < 0 \lor (p_{17} = 0 \lor p_5 = 0) \land$
$(p_{12} p_{17} p_{20} p_3 < 0 \lor p_{11} p_{17} p_6 < 0 \land (p_{17} = 0 \lor p_3 = 0))))$

where the $p_1, \ldots, p_{26}$ are irreducible polynomials – some of them large

$$
\begin{aligned}
p_1 = {} & \delta^4\gamma_0^2 v_3^{18} + 2\delta^4\gamma_0^2 v_3^9 + \delta^4\gamma_0^2 - 18\delta^4\gamma_0 v_1 v_2 v_3^{26} - 36\delta^4\gamma_0 v_1 v_2 v_3^{17} - 18\delta^4\gamma_0 v_1 v_2 v_3^8 + 16\delta^4\gamma_0 v_1 v_3^{27} + 30\delta^4\gamma_0 v_1 v_3^{18} + \\
& 12\delta^4\gamma_0 v_1 v_3^9 - 2\delta^4\gamma_0 v_1 + 81\delta^4 v_1^2 v_2^2 v_3^{34} + 162\delta^4 v_1^2 v_2^2 v_3^{25} + 81\delta^4 v_1^2 v_2^2 v_3^{16} - 144\delta^4 v_1^2 v_2 v_3^{35} - 270\delta^4 v_1^2 v_2 v_3^{26} - 108\delta^4 v_1^2 v_2 v_3^{17} + \\
& 18\delta^4 v_1^2 v_2 v_3^8 + 64\delta^4 v_1^2 v_3^{36} + 112\delta^4 v_1^2 v_3^{27} + 33\delta^4 v_1^2 v_3^{18} - 14\delta^4 v_1^2 v_3^9 + \delta^4 v_1^2 - 18\delta^3\gamma_0^2\lambda v_1 v_3^{17} - 18\delta^3\gamma_0^2\lambda v_1 v_3^8 + \\
& 162\delta^3\gamma_0\lambda v_1^2 v_2 v_3^{25} + 162\delta^3\gamma_0\lambda v_1^2 v_2 v_3^{16} - 126\delta^3\gamma_0\lambda v_1^2 v_3^{26} - 90\delta^3\gamma_0\lambda v_1^2 v_3^{17} + 36\delta^3\gamma_0\lambda v_1^2 v_3^8 - 162\delta^3\lambda v_1^3 v_2 v_3^{34} - \\
& 324\delta^3\lambda v_1^3 v_2 v_3^{25} - 162\delta^3\lambda v_1^3 v_2 v_3^{16} + 144\delta^3\lambda v_1^3 v_3^{35} + 270\delta^3\lambda v_1^3 v_3^{26} + 108\delta^3\lambda v_1^3 v_3^{17} - 18\delta^3\lambda v_1^3 v_3^8 + 81\delta^2\gamma_0^2\lambda^2 v_1^2 v_3^{16} - \\
& 36\delta^2\gamma_0^2\lambda v_1 v_3^{17} - 36\delta^2\gamma_0^2\lambda v_1 v_3^8 - 2\delta^2\gamma_0^2 v_3^{18} - 4\delta^2\gamma_0^2 v_3^9 - 2\delta^2\gamma_0^2 - 162\delta^2\gamma_0\lambda^2 v_1^3 v_3^{25} - 162\delta^2\gamma_0\lambda^2 v_1^3 v_3^{16} + 324\delta^2\gamma_0\lambda v_1^2 v_2 v_3^{25} + \\
& 324\delta^2\gamma_0\lambda v_1^2 v_2 v_3^{16} - 252\delta^2\gamma_0\lambda v_1^2 v_3^{26} - 180\delta^2\gamma_0\lambda v_1^2 v_3^{17} + 72\delta^2\gamma_0\lambda v_1^2 v_3^8 + 18\delta^2\gamma_0 v_1 v_2 v_3^{26} + 36\delta^2\gamma_0 v_1 v_2 v_3^{17} + \\
& 18\delta^2\gamma_0 v_1 v_2 v_3^8 - 14\delta^2\gamma_0 v_1 v_3^{27} - 24\delta^2\gamma_0 v_1 v_3^{18} - 6\delta^2\gamma_0 v_1 v_3^9 + 4\delta^2\gamma_0 v_1 + 81\delta^2\lambda^2 v_1^4 v_3^{34} + 162\delta^2\lambda^2 v_1^4 v_3^{25} + 81\delta^2\lambda^2 v_1^4 v_3^{16} - \\
& 324\delta^2\lambda v_1^3 v_2 v_3^{34} - 648\delta^2\lambda v_1^3 v_2 v_3^{25} - 324\delta^2\lambda v_1^3 v_2 v_3^{16} + 288\delta^2\lambda v_1^3 v_3^{35} + 540\delta^2\lambda v_1^3 v_3^{26} + 216\delta^2\lambda v_1^3 v_3^{17} - 36\delta^2\lambda v_1^3 v_3^8 - \\
& 18\delta^2 v_1^2 v_2 v_3^{35} - 54\delta^2 v_1^2 v_2 v_3^{26} - 54\delta^2 v_1^2 v_2 v_3^{17} - 18\delta^2 v_1^2 v_2 v_3^8 + 16\delta^2 v_1^2 v_3^{36} + 46\delta^2 v_1^2 v_3^{27} + 42\delta^2 v_1^2 v_3^{18} + 10\delta^2 v_1^2 v_3^9 - 2\delta^2 v_1^2 - \\
& 18\delta\gamma_0^2\lambda v_1 v_3^{17} - 18\delta\gamma_0^2\lambda v_1 v_3^8 + 36\delta\gamma_0\lambda v_1^2 v_3^{26} + 72\delta\gamma_0\lambda v_1^2 v_3^{17} + 36\delta\gamma_0\lambda v_1^2 v_3^8 - 18\delta\lambda v_1^3 v_3^{35} - 54\delta\lambda v_1^3 v_3^{26} - 54\delta\lambda v_1^3 v_3^{17} - \\
& 18\delta\lambda v_1^3 v_3^8 + \gamma_0^2 v_3^{18} + 2\gamma_0^2 v_3^9 + \gamma_0^2 - 2\gamma_0 v_1 v_3^{27} - 6\gamma_0 v_1 v_3^{18} - 6\gamma_0 v_1 v_3^9 - 2\gamma_0 v_1 + v_1^2 v_3^{36} + 4v_1^2 v_3^{27} + 6v_1^2 v_3^{18} + 4v_1^2 v_3^9 + v_1^2
\end{aligned}
$$

$$
\begin{aligned}
p_2 = {} & \delta^4 v_3^{18} + 2\delta^4 v_3^9 + \delta^4 - 18\delta^3\lambda v_1 v_3^{17} - 18\delta^3\lambda v_1 v_3^8 + 81\delta^2\lambda^2 v_1^2 v_3^{16} - 36\delta^2\lambda v_1 v_3^{17} - 36\delta^2\lambda v_1 v_3^8 - 2\delta^2 v_3^{18} - \\
& 4\delta^2 v_3^9 - 2\delta^2 - 18\delta\lambda v_1 v_3^{17} - 18\delta\lambda v_1 v_3^8 + v_3^{18} + 2v_3^9 + 1
\end{aligned}
$$

$$
p_3 = 3\delta^3 v_3^9 + 3\delta^3 - 18\delta^2\lambda v_1 v_3^8 + 8\delta^2 v_3^9 + 8\delta^2 - 9\delta\lambda v_1 v_3^8 + 6\delta v_3^9 + 6\delta + v_3^9 + 1
$$

$$
\begin{aligned}
p_4 = {} & \delta^2\gamma_0 v_3^9 + \delta^2\gamma_0 - 9\delta^2 v_1 v_2 v_3^{17} - 9\delta^2 v_1 v_2 v_3^8 + 8\delta^2 v_1 v_3^{18} + 7\delta^2 v_1 v_3^9 - \delta^2 v_1 - 9\delta\gamma_0\lambda v_1 v_3^8 + 9\delta\lambda v_1^2 v_3^{17} + 9\delta\lambda v_1^2 v_3^8 - \\
& 18\gamma_0\lambda v_1 v_3^8 - \gamma_0 v_3^9 - \gamma_0 + 18\lambda v_1^2 v_3^{17} + 18\lambda v_1^2 v_3^8 + v_1 v_3^{18} + 2v_1 v_3^9 + v_1
\end{aligned}
$$

$$
p_5 = 3\delta^2 v_3^9 + 3\delta^2 - 9\delta\lambda v_1 v_3^8 + 7\delta v_3^9 + 7\delta + 3v_3^9 + 3
$$

$$
p_6 = \delta^2 v_3^9 + \delta^2 - 9\delta\lambda v_1 v_3^8 + 2\delta v_3^9 + 2\delta + v_3^9 + 1
$$

$$
p_7 = \delta^2 v_3^9 + \delta^2 - 9\delta\lambda v_1 v_3^8 - 18\lambda v_1 v_3^8 - v_3^9 - 1
$$

$$
p_8 = 9\delta\lambda v_1 v_3^8 + \delta v_3^9 + \delta + v_3^9 + 1
$$

$p_9 = 9\delta\lambda v_1 v_3^8 - v_3^9 - 1$

$p_{10} = \delta + 2$

$p_{11} = \delta + 1$

$p_{12} = \delta$, $p_{13} = \gamma_0\mu + \lambda v_1 - v_2$

$p_{14} = \gamma_0 - v_1 v_3^9 - v_1$, $p_{15} = \gamma_0$

$p_{16} = 18\lambda v_1 v_3^8 + v_3^9 + 1$, $p_{17} = 9\lambda v_1 v_3^8 + v_3^9 + 1$

$p_{18} = \lambda$

$p_{19} = \mu$

$p_{20} = v_1$

$p_{21} = v_2 - v_3$

$p_{22} = v_2$

$p_{23} = v_3^6 - v_3^3 + 1$

$p_{24} = v_3^2 - v_3 + 1$

$p_{25} = v_3 + 1$

$p_{26} = v_3$

# Hopf Bifurcations in System Biology and Chemistry

[AB 2008, Math Comput Sci 2009, Bull Math Biol 2011, J Comput Phys 2016]

Some results and times:

| $n$ | $\exists\varphi_n$ | $\exists\varphi_n[\lambda \leftarrow -\lambda]$ | $\exists\varphi_n[\lambda \leftarrow 0]$ | time (s) |
|---|---|---|---|---|
| 2 | false | false | false | $< 0.01$ |
| 3 | false | false | false | 19.28 |
| 4 | false | false | false | 21.58 |
| 5 | false | false | false | 19.09 |
| 6 | false | false | false | 23.72 |
| 7 | false | false | false | 23.89 |
| 8 | false | false | false | 22.35 |
| 9 | true | false | false | 0.17 |
| 10 | true | false | false | 0.17 |

Real quantifier elimination also delivers sample solutions, e.g., for $n = 9$:

$\alpha = 1$ $\quad\delta = 1$ $\quad\gamma_0 = 0.0100554964908$

$\lambda = 17617230.5528$ $\quad\mu = 0$ $\quad\vartheta = 0.0000211443608455$

$v_1 = 0.000000170287832189$ $\quad v_2 = 3$ $\quad v_3 = 1.24573093962$

# SYMBIONT PROJECT

- French–German project 2018–2021 (ANR-17-CE40-0036 and DFG-391322026)
- Montpellier, Lille, Nancy, Saclay; Aachen, Bonn, Kassel
- Interdisciplinary: 1/3 each of researchers from biology, informatics, mathematics
- We are hiring PhD students and Postdocs!
- www.symbiont-project.org

# Profiling $\varphi_9$

| Toplevel Calls | calls | time(ms) | time(%) | gc(ms) | gc(%) |
|---|---|---|---|---|---|
| sfto_fctrf | 22590 | 44560 | 44.2 | 8850 | 37.9 |
| sfto_gcdf | 1024648 | 42770 | 42.4 | 11530 | 49.4 |
| ofsf_qesubcrme2 | 12 | 4050 | 4.0 | 1100 | 4.7 |
| ofsf_qesubcqme | 80 | 2890 | 2.8 | 620 | 2.6 |
| ofsf_qesubcq | 105 | 1411 | 1.4 | 271 | 1.1 |
| ofsf_qesubcr2 | 7 | 350 | 0.3 | 40 | 0.1 |
| ofsf_qesubcr1 | 10 | 79 | 0.0 | 0 | 0.0 |
| ofsf_qesubcrme1 | 6 | 40 | 0.0 | 10 | 0.0 |
| ofsf_qesubi | 17 | 10 | 0.0 | 0 | 0.0 |
| sum | 1047475 | 96160 | 95.1 | 22421 | 95.8 |
| total | | 100650 | 100.0 | 23310 | 100.0 |

| | | | |
|---|---|---|---|
| virtual substitutions | 237 | $\approx$ | $10^2$ |
| polynomial factorizations | 22590 | $\approx$ | $10^4$ |
| GCD computations | 1024648 | $\approx$ | $10^6$ |

Examples that are up to 5 orders of magnitude larger are realistic at present.

# What Else Have We Got?

**Our online database REMIS** at `www.redlog.eu/remis/`

- contains $10^2$ examples (101, 77 of them real)
- not updated recently; number of real examples could be doubled

**SMT benchmarks** for QF_LRA, LRA, QF_NRA, NRA

- $10^4$ to $10^5$ examples
- quite redundant
- no parameters
- mostly existential

# Answers and Questions

**Answers**

- Reduce/Redlog is a Lisp system.
- We can easily overload functions at runtime.
- Try several options and dump for learning
- Yes, if any this works, then there will many more spots – I only gave some examples.

**Questions**

- Have we got enough examples?
- How to learn about formulas, which are not real vectors?
- How to cover logical structure, relations, algebraic structure degrees?
- Do I have to tell you what are good intermediate results, or is sufficient to consider final success?
- I answered above how to learn. But then, what do I have to do in Redlog when learning is done?