Machine Learning for Automated Reasoning

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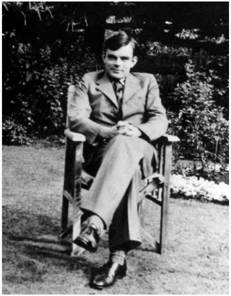
Induction/Learning vs Reasoning – Henri Poincaré

- Science and Method: Ideas about the interplay between correct deduction and induction/intuition
- *“And in demonstration itself logic is not all. The true mathematical reasoning is a real induction [...]”*
- I believe he was right: strong general reasoning engines have to combine deduction and induction (learning patterns from data, making conjectures, etc.)

Learning vs Reasoning – Alan Turing 1950 – AI



- 1950: *Computing machinery and intelligence* – AI, Turing test
- *"We may hope that machines will eventually compete with men in all purely intellectual fields."* (regardless of his 1936 undecidability result!)
- last section on Learning Machines(!):
- *"But which are the best ones [fields] to start [learning on] with?"*
- *"... Even this is a difficult decision. Many people think that a very abstract activity, like the playing of chess, would be best."*
- Why not try with large computer-understandable math corpora?
- (... I have been trying since my MSc work)

Using Learning to Guide Theorem Proving

- **high-level**: pre-select lemmas from a large library, give them to ATPs
- **high-level**: pre-select a good ATP strategy/portfolio for a problem
- **high-level**: pre-select good *hints* for a problem, use them to guide ATPs
- **low-level**: guide every inference step of ATPs (tableau, superposition)
- **low-level**: guide every kernel step of LCF-style ITPs
- **mid-level**: guide application of tactics in ITPs
- **mid-level**: invent suitable ATP strategies for classes of problems
- **mid-level**: invent suitable conjectures for a problem
- **mid-level**: invent suitable concepts/models for problems/theories
- **proof sketches**: explore stronger/related theories to get proof ideas
- **theory exploration**: develop interesting theories by conjecturing/proving
- **feedback loops**: (dis)prove, learn from it, (dis)prove more, learn more, ...
- ...

Sample of Learning Approaches We Have Been Using

- **neural networks** (statistical ML) – backpropagation, deep learning, convolutional, recurrent, etc.
- **decision trees, random forests, gradient tree boosting** – find good classifying attributes (and/or their values); more explainable
- **support vector machines** – find a good classifying hyperplane, possibly after non-linear transformation of the data (*kernel methods*)
- **k-nearest neighbor** – find the *k* nearest neighbors to the query, combine their solutions
- **naive Bayes** – compute probabilities of outcomes assuming complete (naive) independence of characterizing features (just multiplying probabilities)
- **inductive logic programming** (symbolic ML) – generate logical explanation (program) from a set of ground clauses by generalization
- **genetic algorithms** – evolve large population by crossover and mutation
- combinations of statistical and symbolic approaches (probabilistic grammars, semantic features, ...)
- supervised, unsupervised, reinforcement learning (actions, explore/exploit, cumulative reward)

Learning – Features and Data Preprocessing

- Extremely important - if irrelevant, there is no use to learn the function from input to output (“garbage in garbage out”)
- Feature discovery – a big field
- Deep Learning – design neural architectures that automatically find important high-level features for a task
- Latent Semantics, dimensionality reduction: use linear algebra (eigenvector decomposition) to discover the most similar features, make approximate equivalence classes from them
- word2vec and related methods: represent words/sentences by *embeddings* (in a high-dimensional real vector space) learned by predicting the next word on a large corpus like Wikipedia
- math and theorem proving: syntactic/semantic patterns/abstractions
- how do we represent math objects (formulas, proofs, ideas) in our mind?

Reasoning Datasets - Large ITP Libraries and Projects

- Mizar / MML / MPTP – since 2003
- MPTP Challenge (2006), MPTP2078 (2011), Mizar40 (2013)
- Isabelle (and AFP) – since 2005
- Flyspeck (including core HOL Light and Multivariate) – since 2012
- HOLStep – 2016, kernel inferences
- Coq – since 2013/2016
- HOL4 – since 2014
- ACL2 – 2014?
- Lean? – 2017?
- Stacks?, ProofWiki?, Arxiv?

Statistical Guidance of Connection Tableau

- learn guidance of every clausal inference in connection tableau (leanCoP)
- set of first-order clauses, *extension* and *reduction* steps
- proof finished when all branches are closed
- a lot of nondeterminism, requires backtracking
- *Iterative deepening* used in leanCoP to ensure completeness
- good for learning – the tableau compactly represents the proof state

Clauses:

$c_1 : P(x)$

$c_2 : R(x, y) \lor \neg P(x) \lor Q(y)$

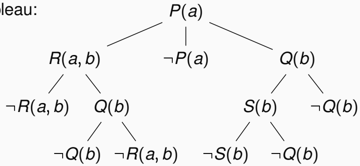
$c_3 : S(x) \lor \neg Q(b)$

$c_4 : \neg S(x) \lor \neg Q(x)$

$c_5 : \neg Q(x) \lor \neg R(a, x)$

$c_6 : \neg R(a, x) \lor Q(x)$

Closed Connection Tableau:



Statistical Guidance of Connection Tableau

- **MaLeCoP** (2011): first prototype Machine Learning Connection Prover
- extension rules chosen by naive Bayes trained on good decisions
- training examples: tableau features plus the name of the chosen clause
- initially slow: off-the-shelf learner 1000 times slower than raw leanCoP
- 20-time search shortening on the MPTP Challenge
- second version: 2015, with C. Kaliszyk
- both prover and naive Bayes in OCAML, fast indexing
- Fairly Efficient MaLeCoP = **FEMaLeCoP**
- 15% improvement over untrained leanCoP on the MPTP2078 problems
- using iterative deepening - enumerate shorter proofs before longer ones

Statistical Guidance of Connection Tableau – rlCoP

- 2018: stronger learners via C interface to OCAML (boosted trees)
- remove iterative deepening, the prover can go arbitrarily deep
- added Monte-Carlo Tree Search (MCTS)
- MCTS search nodes are sequences of clause application
- a good heuristic to explore new vs exploit good nodes:

$$\frac{w_i}{n_i} + c \cdot p_i \cdot \sqrt{\frac{\ln N}{n_i}} \qquad \text{(UCT - Kocsis, Szepesvari 2006)}$$

- learning both *policy* (clause selection) and *value* (state evaluation)
- clauses represented not by names but also by features (generalize!)
- binary learning setting used: | proof state | clause features |
- mostly term walks of length 3 (trigrams), hashed into small integers
- many iterations of proving and learning

Statistical Guidance of Connection Tableau – rlCoP

- On 32k Mizar40 problems using 200k inference limit
- nonlearning CoPs:

System	leanCoP	bare prover	rlCoP no policy/value (UCT only)
Training problems proved	10438	4184	7348
Testing problems proved	**1143**	431	804
Total problems proved	11581	4615	8152

- rlCoP with policy/value after 5 proving/learning iters on the training data
- $1624/1143 = 42.1$% improvement over leanCoP on the testing problems

Iteration	1	2	3	4	5	6	7	8
Training proved	12325	13749	14155	14363	14403	14431	14342	**14498**
Testing proved	1354	1519	1566	1595	**1624**	1586	1582	1591

Statistical Guidance the Given Clause in E Prover

- harder for learning than tableau
- the proof state are two large heaps of clauses *processed*/*unprocessed*
- 2017: ENIGMA (features engineering), Deep guidance (neural nets)
- both learn on E's proof search traces, put classifier in E
- positive examples: given clauses used in the proof
- negative examples: given clauses not used in the proof
- ENIGMA: fast feature extraction followed by fast/sparse linear classifier
- about 80% improvement on the AIM benchmark
- Deep guidance: convolutional nets - no feature engineering but slow

ProofWatch: Statistical/Semantic Guidance of E

- Bob Veroff's *hints* method used for Prover9/AIM
- solve many easier/related problems
- load their useful lemmas on the *watchlist*
- boost inferences on clauses that subsume a watchlist clause
- watchlist parts are fast thinking, bridged by standard search
- ProofWatch (2018): load many proofs separately
- dynamically boost those that have been covered more
- needed for heterogeneous ITP libraries
- statistical: watchlists chosen using similarity and usefulness
- semantic/deductive: dynamic guidance based on exact proof matching
- results in better vectorial characterization of saturation proof searches

ProofWatch: Statistical/Symbolic Guidance of E

```
theorem Th36: :: YELLOW_5:36
for L being non empty Boolean RelStr for a, b being Element of L
holds ( 'not' (a "\/" b) = ('not' a) "/\" ('not' b)
         & 'not' (a "/\" b) = ('not' a) "\/" ('not' b) )
```

- De Morgan's laws for Boolean lattices
- guided by 32 related proofs resulting in 2220 watchlist clauses
- 5218 given clause loops, resulting ATP proof is 436 clauses
- 194 given clauses match the watchlist and 120 (61.8%) used in the proof
- most helped by the proof of WAYBEL_1:85 – done for lower-bounded Heyting

```
theorem :: WAYBEL_1:85
for H being non empty lower-bounded RelStr st H is Heyting holds
for a, b being Element of H holds
  'not' (a "/\" b) >= ('not' a) "\/" ('not' b)
```

ProofWatch: Vectorial Proof State

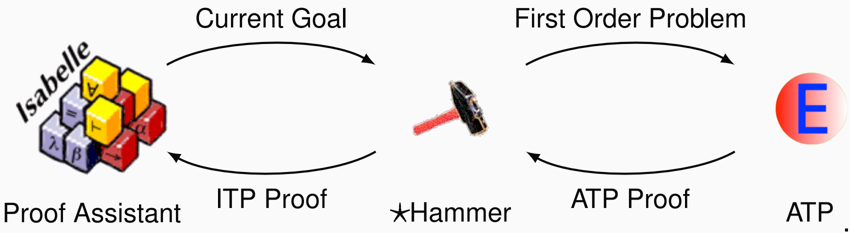
Final state of the proof progress for the 32 proofs guiding `YELLOW_5:36`

0	0.438	42/96	1	0.727	56/77	2	0.865	45/52	3	0.360	9/25
4	0.750	51/68	5	0.259	7/27	6	0.805	62/77	7	0.302	73/242
8	0.652	15/23	9	0.286	8/28	10	0.259	7/27	11	0.338	24/71
12	0.680	17/25	13	0.509	27/53	14	0.357	10/28	15	0.568	25/44
16	0.703	52/74	17	0.029	8/272	18	0.379	33/87	19	0.424	14/33
20	0.471	16/34	21	0.323	20/62	22	0.333	7/21	23	0.520	26/50
24	0.524	22/42	25	0.523	45/86	26	0.462	6/13	27	0.370	20/54
28	0.411	30/73	29	0.364	20/55	30	0.571	16/28	31	0.357	10/28

High-level ATP guidance: Premise Selection/Hammers

- 2003: Can existing ATPs be used on the freshly translated Mizar library?
- About 80000 nontrivial math facts at that time – impossible to use them all
- Mizar Proof Advisor (2003):
- train naive-Bayes fact selection on previous Mizar/MML
- recommend relevant premises when proving new conjectures
- give them to unmodified FOL ATPs
- possibly reconstruct inside the ITP afterwards (lots of work)
- First results over the whole Mizar library in 2003:
  - about 70% coverage in the first 100 recommended premises
  - chain the recommendations with strong ATPs to get full proofs
  - about 14% of the Mizar theorems were then automatically provable (SPASS)

Today's AI-ATP systems (⋆-Hammers)



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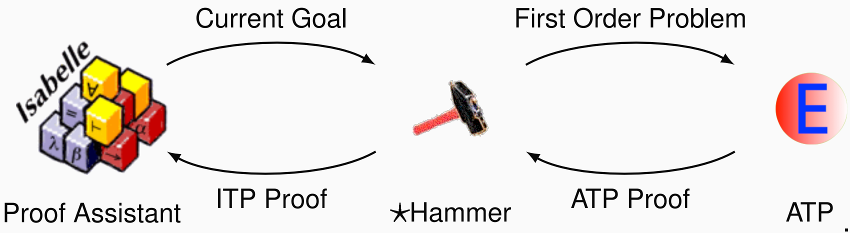
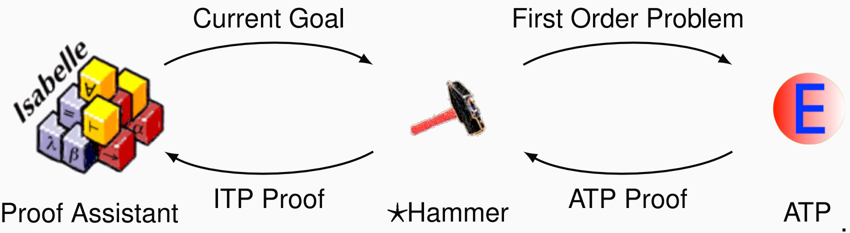


How much can it do?

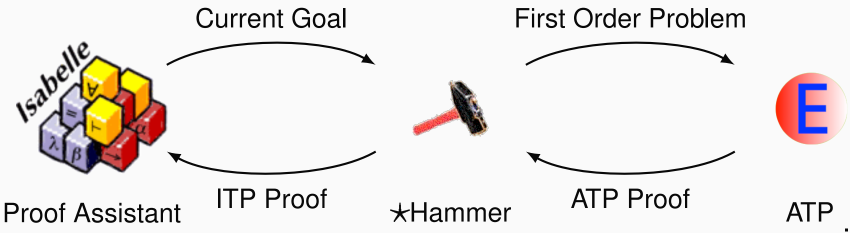
Today's AI-ATP systems (⋆-Hammers)



How much can it do?

- Mizar / MML – MizAR
- Isabelle (Auth, Jinja) – Sledgehammer
- Flyspeck (including core HOL Light and Multivariate) – HOL(y)Hammer
- HOL4 (Gauthier and Kaliszyk)
- CoqHammer (Czajka and Kaliszyk) - about 40% on Coq standard library

Today's AI-ATP systems ($\star$-Hammers)



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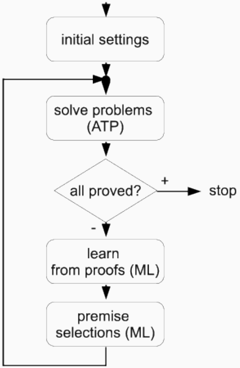
$\approx$ 45% success rate

Machine Learner for Automated Reasoning

- MaLARea (2006) – infinite hammering
- feedback loop interleaving ATP with learning premise selection
- both syntactic and semantic features for characterizing formulas:
- evolving set of finite (counter)models in which formulas evaluated



Recent Improvements and Additions

- Semantic features encoding term matching/unification [IJCAI'15]
- Distance-weighted k-nearest neighbor, LSI, boosted trees (XGBoost)
- Matching and transferring concepts and theorems between libraries (Gauthier & Kaliszyk) – allows "superhammers", conjecturing, and more
- Lemmatization – extracting and considering millions of low-level lemmas
- First useful CoqHammer (Czajka & Kaliszyk 2016), 40%–50% reconstruction/ATP success on the Coq standard library
- Neural sequence models, definitional embeddings (Google Research)
- Hammers combined with statistical tactical search: TacticToe (HOL4)
- Learning in binary setting from many alternative proofs
- Negative/positive mining (ATPBoost)

Summary of Features Used

- From syntactic to more semantic:
- Constant and function symbols
- Walks in the term graph
- Walks in clauses with polarity and variables/skolems unified
- Subterms, de Bruijn normalized
- Subterms, all variables unified
- Matching terms, no generalizations
- terms and (some of) their generalizations
- Substitution tree nodes
- All unifying terms
- Evaluation in a large set of (finite) models
- LSI/PCA combinations of above
- Neural embeddings of above

TacticToe: mid-level ITP Guidance (Gauthier et al.)

- learns from human tactical HOL4 proofs to solve new goals
- no translation or reconstruction needed
- similar to rlCoP: policy/value learning
- however much more technically challenging:
    - tactic and goal state recording
    - tactic argument abstraction
    - absolutization of tactic names
    - nontrivial evaluation issues
- policy: which tactic/parameters to choose for a current goal?
- value: how likely is this proof state succeed?
- 66% of HOL4 toplevel proofs in 60s (better than a hammer!)
- work in progress for Coq
- earlier Coq work: SEPIA (Gransden et al, 2015) - inferred automata

Neural Autoformalization (Wang et al., 2018)

- generate about 1M Latex - Mizar pairs based on Bancerek's work
- train neural seq-to-seq translation models (Luong – NMT)
- evaluate on about 100k examples
- many architectures tested, some work much better than others
- very important latest invention: *attention* in the seq-to-seq models
- more data very important for neural training – our biggest bottleneck (you can help!)

Neural Autoformalization data

Rendered LaTeX	If $X \subseteq Y \subseteq Z$, then $X \subseteq Z$.
Mizar	`X c= Y & Y c= Z implies X c= Z;`
Tokenized Mizar	`X c= Y & Y c= Z implies X c= Z ;`
LaTeX	`If $X \subseteq Y \subseteq Z$, then $X \subseteq Z$.`
Tokenized LaTeX	`If $ X \subseteq Y \subseteq Z $ , then $ X \subseteq Z $ .`

Neural Autoformalization results

Parameter	Final Test Perplexity	Final Test BLEU	Identical Statements (%)	Identical No-overlap (%)
128 Units	3.06	41.1	40121 (38.12%)	6458 (13.43%)
256 Units	1.59	64.2	63433 (60.27%)	19685 (40.92%)
512 Units	1.6	**67.9**	66361 (63.05%)	21506 (44.71%)
1024 Units	**1.51**	61.6	**69179 (65.73%)**	**22978 (47.77%)**
2048 Units	2.02	60	59637 (56.66%)	16284 (33.85%)

Neural Fun – Performance after Some Training

Rendered LaTeX	Suppose s_8 is convergent and s_7 is convergent . Then $\mathop{\rm lim}(s_8{+}s_7) \mathrel{=} \mathop{\rm lim} s_8{+} \mathop{\rm lim} s_7$
Input LaTeX	`Suppose $ { s _ { 8 } } $ is convergent and $ { s _ { 7 } } $ is convergent . Then $ \mathop { \rm lim } ( { s _ { 8 } } { + } { s _ { 7 } } } ) \mathrel { = } \mathop { \rm lim } { s _ { 8 } } { + } \mathop { \rm lim } { s _ { 7 } } $ .`
Correct	`seq1 is convergent & seq2 is convergent implies lim ( seq1 + seq2 ) = ( lim seq1 ) + ( lim seq2 ) ;`
Snapshot-1000	`x in dom f implies ( x * y ) * ( f | ( x | ( y | ( y | y ) ) ) ) = ( x | ( y | ( y | ( y | y ) ) ) ) ) ;`
Snapshot-2000	`seq is summable implies seq is summable ;`
Snapshot-3000	`seq is convergent & lim seq = 0c implies seq = seq ;`
Snapshot-4000	`seq is convergent & lim seq = lim seq implies seq1 + seq2 is convergent ;`
Snapshot-5000	`seq1 is convergent & lim seq2 = lim seq2 implies lim_inf seq1 = lim_inf seq2 ;`
Snapshot-6000	`seq is convergent & lim seq = lim seq implies seq1 + seq2 is convergent ;`
Snapshot-7000	`seq is convergent & seq9 is convergent implies lim ( seq + seq9 ) = ( lim seq ) + ( lim seq9 ) ;`

Some References

- C. Kaliszyk, J. Urban, H. Michalewski, M. Olsak: Reinforcement Learning of Theorem Proving. CoRR abs/1805.07563 (2018)
- Z. Goertzel, J. Jakubuv, S. Schulz, J. Urban: ProofWatch: Watchlist Guidance for Large Theories in E. CoRR abs/1802.04007 (2018)
- T. Gauthier, C. Kaliszyk, J. Urban, R. Kumar, M. Norrish: Learning to Prove with Tactics. CoRR abs/1804.00596 (2018).
- J. Jakubuv, J. Urban: ENIGMA: Efficient Learning-Based Inference Guiding Machine. CICM 2017: 292-302
- S. M. Loos, G. Irving, C. Szegedy, C. Kaliszyk: Deep Network Guided Proof Search. LPAR 2017: 85-105
- L. Czajka, C. Kaliszyk: Hammer for Coq: Automation for Dependent Type Theory. J. Autom. Reasoning 61(1-4): 423-453 (2018)
- J. C. Blanchette, C. Kaliszyk, L. C. Paulson, J. Urban: Hammering towards QED. J. Formalized Reasoning 9(1): 101-148 (2016)
- G. Irving, C. Szegedy, A. Alemi, N. Eén, F. Chollet, J. Urban: DeepMath - Deep Sequence Models for Premise Selection. NIPS 2016: 2235-2243
- C. Kaliszyk, J. Urban, J. Vyskocil: Efficient Semantic Features for Automated Reasoning over Large Theories. IJCAI 2015: 3084-3090
- J. Urban, G. Sutcliffe, P. Pudlák, J. Vyskocil: MaLARea SG1- Machine Learner for Automated Reasoning with Semantic Guidance. IJCAR 2008: 441-456
- C. Kaliszyk, J. Urban, J. Vyskocil: Automating Formalization by Statistical and Semantic Parsing of Mathematics. ITP 2017: 12-27
- Q. Wang, C. Kaliszyk, J. Urban: First Experiments with Neural Translation of Informal to Formal Mathematics. CoRR abs/1805.06502 (2018)
- J. Urban, J. Vyskocil: Theorem Proving in Large Formal Mathematics as an Emerging AI Field. LNCS 7788, 240-257, 2013.

Thanks and Advertisement

- Thanks for your attention!
- AITP – Artificial Intelligence and Theorem Proving
- April 8–12, 2019, Obergurgl, Austria, `aitp-conference.org`
- ATP/ITP/Math vs AI/Machine-Learning people, Computational linguists
- Discussion-oriented and experimental
- Grown to 60 people in 2018