In a search for cheaper computer algebra tools to answer real world problems

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Math CU Seminar

EPSRC grant EP/T015748/1 (The DEWCAD Project)
Hungarian grant NKFIH KKP 129877

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- What is Computer Algebra?
- Why Computer Algebra?
- A question from Population Dynamics.
- Asking help from Computer Algebra.
- Limitations of Computer Algebra.
- Asking help from humans :)

What is Computer Algebra?

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Computer = คอมพิวเตอร์

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Example

If the length of an edge of a square shape land is 400 meters, then its area is 1 rai.

ถ้าความยาวของที่ดินรูปสี่เหลี่ยมจัตุรัส
เท่ากับ ๔๐๐ เมตร พื้นที่ของที่ดินนั้นเท่ากับ
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Example

If the length of an edge of a square shape land is 800 meters, then its area is 4 rai.

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Example

If the length of an edge of a square shape land is 100 meters, then its area is $\frac{1}{16}$ rai.

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Example

If the length of an edge of a square shape land is x meters, then its area is $\frac{1}{16}x^2$ rai.

ถ้าความยาวของที่ดินรูปสี่เหลี่ยมจัตุรัสเท่ากับ x เมตร พื้นที่ของที่ดินนั้นเท่ากับ $\frac{1}{1600}x^2$ ไร่

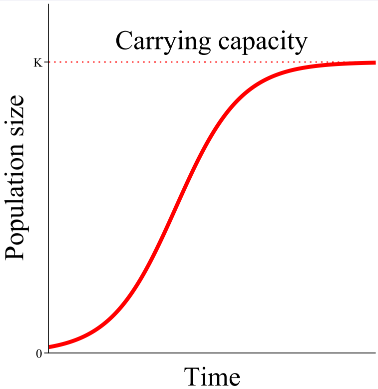
Why Computer Algebra?

Few examples of questions that computer algebra can solve:

- Graph coloring,
- Hypergraph coloring,
- Feasibility of nonlinear programming problems,
- Finding extreme points of a convex set,
- Quantifier elimination in nonlinear real arithmetic logic,
- Finding bifurcations of a nonlinear ODE system.

Population growth

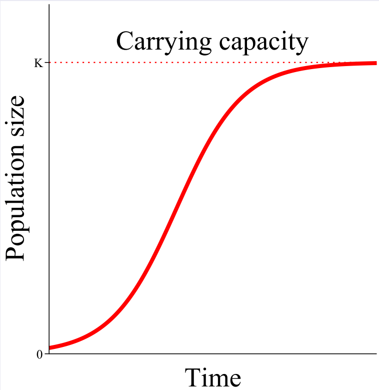
Logistic growth



$$\dot{N} = rN(1 - \frac{N}{K})$$

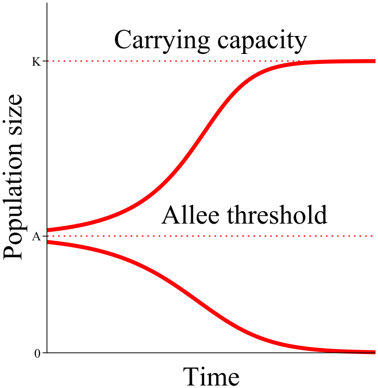
Population growth

Logistic growth



$$\dot{N} = rN(1 - \frac{N}{K})$$

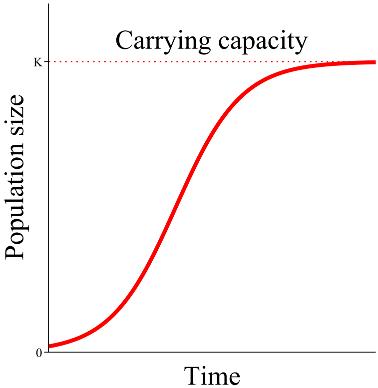
(Strong) Allee effect



$$\dot{N} = rN(1 - \frac{N}{K})(\frac{N}{A} - 1)$$

Population growth

Logistic growth



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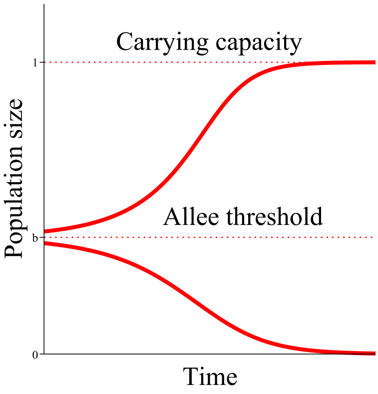
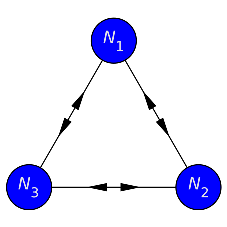


$$\dot{N} = N(1 - N)(N - b)$$

Main model

n-connected populations with Allee effect.

$$\dot{N}_i = N_i(1 - N_i)(N_i - b) - (n-1)aN_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} aN_j, \quad i = 1, \cdots, n.$$



Main model

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$$N_i(1 - N_i)(N_i - b) - (n - 1)aN_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} aN_j = 0, \quad i = 1, \cdots, n.$$

This parametric system has n variables, N_is, and 2 parameters, the Allee effect parameter b and the strength of connectivity a.

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The model has 3^n non-negative steady states for small a and 3 for large a.

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The model has 3^n non-negative steady states for small a and 3 for large a.

Questions

- What happens for not very small and not very large values of a?
- Is it true that the number of non-negative solutions is non-increasing with respect to a?

Approach no. 1

Original system

$\{x^2 + bx + c = 0\}$

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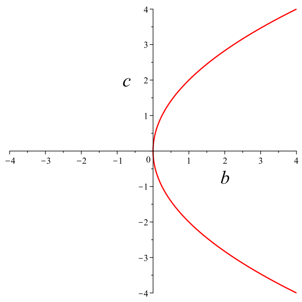
$\{x^2 + bx + c = 0\}$

$\Downarrow$ Discriminant Variety (Elimination via Gröbner bases).

$\{b^2 - 4c\}$.



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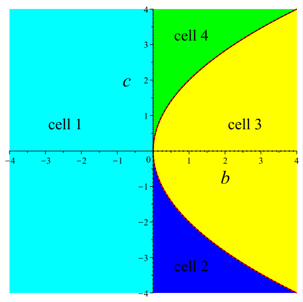

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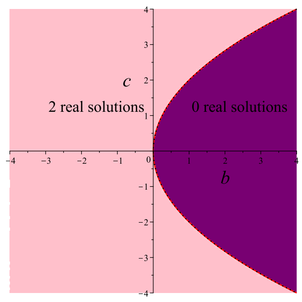
$\#(f^{-1}(0) \cap \mathbb{R}) =$

$$\begin{cases} 2 & ; (c, b) \in \text{cells } 1,2,4 \\ 0 & ; (c, b) \in \text{cell } 3 \end{cases}$$



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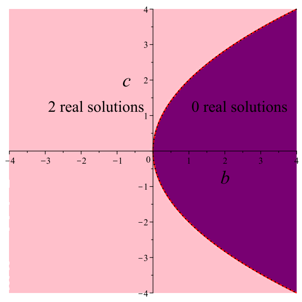
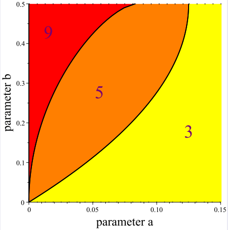


Open CAD with respect to the discriminant variety is already implemented in a Maple package.

Using approach 1 (used in ref. 3)

Recall the *n*-patches model.

Maple* can compute the 2-dimensional CAD* of the model for $n = 2$.



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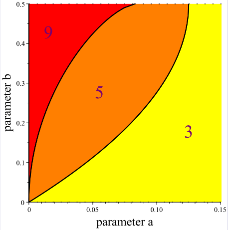
Recall the *n*-patches model.

Maple* can compute the 2-dimensional CAD* of the model for $n = 2$. But not for $n = 3$.



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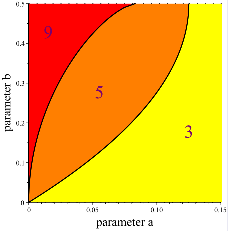

- Why?
  The complexity of this algorithm is doubly exponential!
- What is doubly exponential complexity?
  See `shorturl.at/bfvyI`

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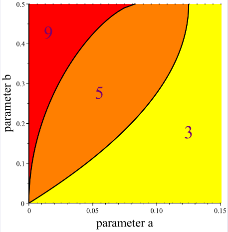
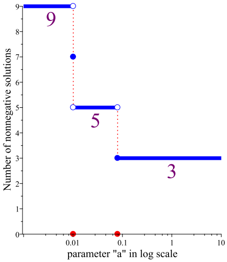


Let's take a step back.

By fixing the value of b, Maple* can compute the 1-dimensional CAD* of the model for $n = 2, 3, 4$, but not $n = 5$.

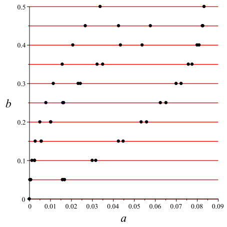
As an example the case $n = 2$ for $b = 0.2$ is shown here.



Approach no. 2 (used in ref. 2)

Using a numeric search.

Finding sections of the 2-dimensional CAD using 1-dimensional CADs for some finite samples of a parameter.



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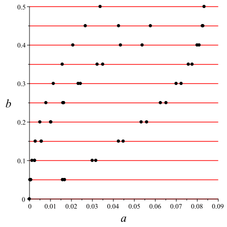
Then using a numeric search to find where the behavior changes.



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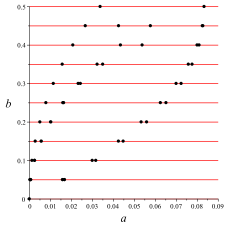
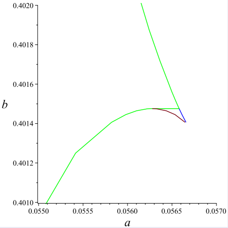
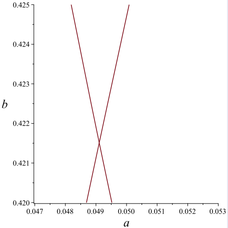
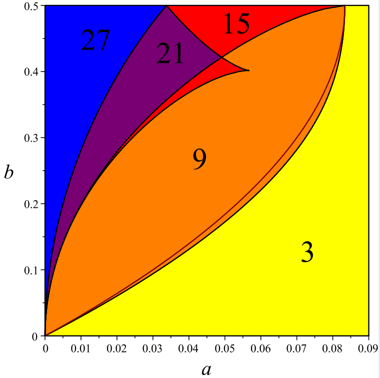


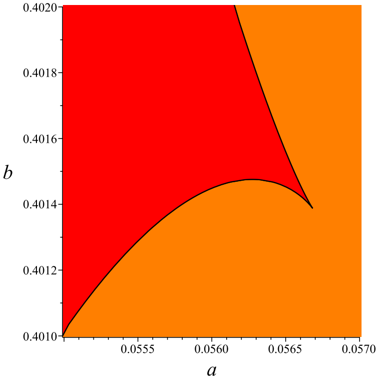




Approach no. 2 (used in ref. 2)

Up to 7 digits accuracy after the decimal point





A discovery

The number of steady states is not always decreasing monotonically by increasing *a*.

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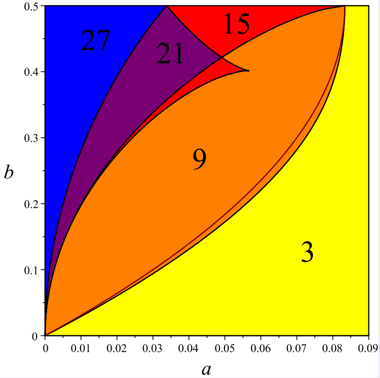

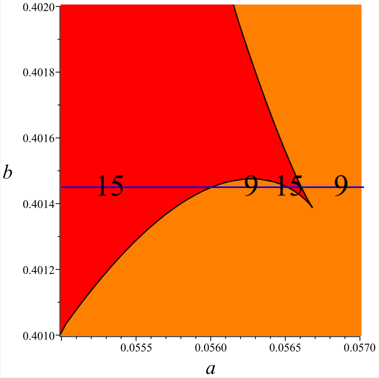

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Further investigation on approach no. 1

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   Computation of the required Gröbner basis for the discriminant variety of approach 1 is not feasible on our computer.
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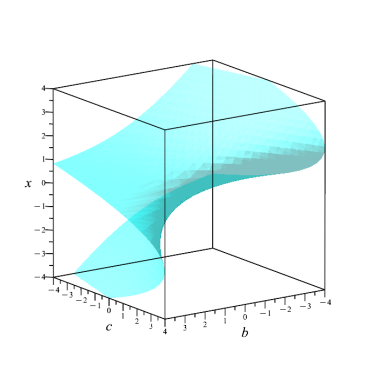
   Yes, using resultant techniques.

What is resultant?

Simple resultant

It receives 2 equations in n variables and returns 1 equation in $n-1$ variables.

$x^2 + bx + c = 0$



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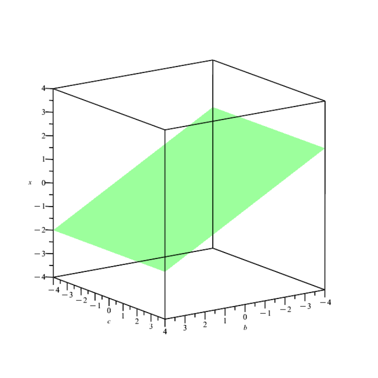

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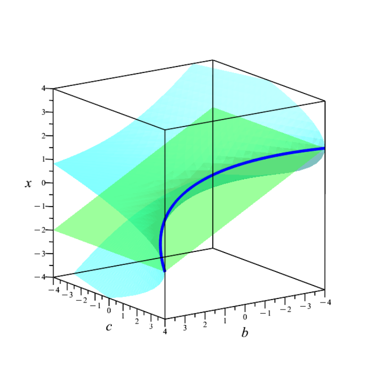

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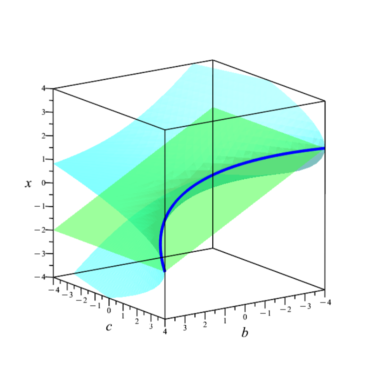
$\mathrm{res_x}(\mathrm{x}^2 + \mathrm{bx} + \mathrm{c}, 2\mathrm{x} + \mathrm{b}) = \mathrm{b}^2 - 4\mathrm{c}$



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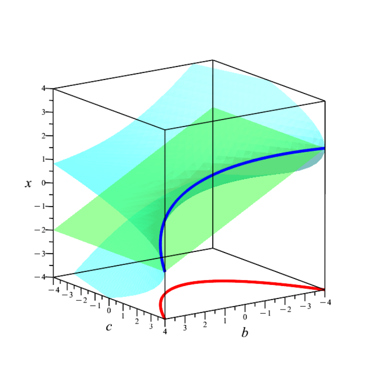
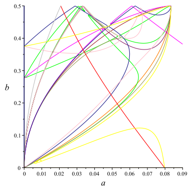
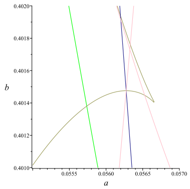
$b^2 - 4c = 0$



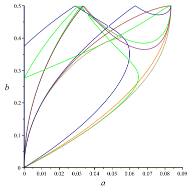
Approach no. 3 (used in ref. 1)

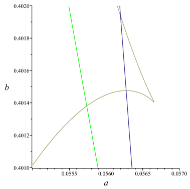
`ResChainSimple`
0.5 second



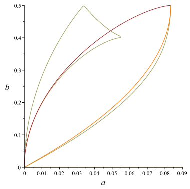


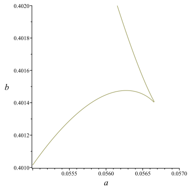
`ResChainBranching`
5 milliseconds





Dixon resultant
7 minutes





Approach no. 4 (used in ref. 1)

1. Dixon resultant has lower worst case complexity than Gröbner basis.

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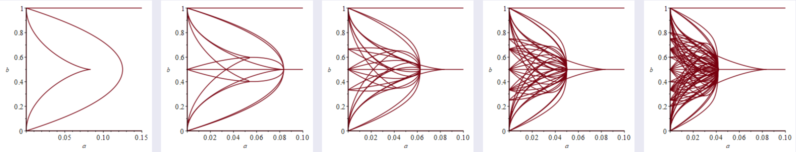

For $n = 6$ it takes 118 seconds. For $n = 7$ we get a Maple error message.

What is new?

We implemented new algorithms in Maple that can handle larger size examples of parametric system of equations (and inequalities) with the following properties;

1. They are free of Gröbner bases computation.
2. They are free of numeric approximations.

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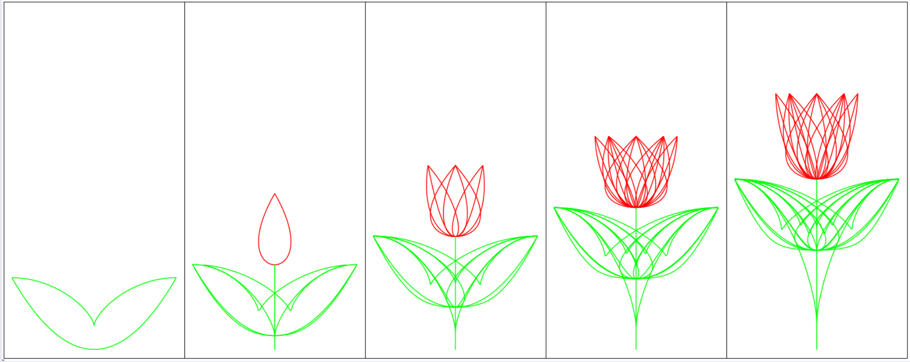
2. They are free of numeric approximations.

Does it mean that the former numerical approach (approach 1) is not interesting anymore?
No, one still can equip approach 2 on top of any of the other approaches and go even further in the size of examples that can be handled by a normal computer.

If you feel artistic

Growing Allee flowers



Displayed at Maple art gallery 2022.

References

1. AmirHosein Sadeghimanesh, Matthew England, *Resultant tools for Parametric Polynomial Systems with Application to Population Models*, **in preparation**, 2022.
2. Gergely Röst, AmirHosein Sadeghimanesh, *Exotic bifurcations in three connected populations with Allee effects*, International Journal of Bifurcation and Chaos, Vol. 31, 2021, DOI: 10.1142/S0218127421502023.
3. Gergely Röst, AmirHosein Sadeghimanesh, *Unidirectional migration of populations with Allee effect*, Letters in Biomathematics, **in press**, 2022.

Thank you for listening.

al-Khwarizmi